CHAPTER 3

Political Equality in Electoral Systems: Equality Implies Proportionality

This chapter argues that the value of political equality implies proportionality in an electoral system. The literature on electoral systems and democratic theory has been remarkably agnostic about how basic values may be translated into institutions. Dahl (1956) constructs an axiomatic theory of democracy based on the idea of political equality but then declares it has little to say about practical politics because it does not consider representative elections. Much of the empirical literature on electoral systems is essentially instrumental—if you want this value, choose this kind of institution. Proportionality is considered as a value but is not grounded in any more fundamental principle. Social choice theory offers a means of translating values into institutions, but this literature has tended to concentrate on social decision rules rather than electoral systems. Here, by contrast, it is argued that proportionality is logically implied by the basic value of political equality, that is, by the concept of democracy itself. Of course, there are many different ways to implement proportionality, and these will be considered later in this chapter. Nevertheless, the value of political equality implies that proportionality is a basic requirement for an electoral system being democratic. If we wish to defend a system that is not proportional, we have to argue that there are other values that outweigh the value of political equality.

This chapter shows that proportionality follows logically from the liberal conception of political equality defined in the last chapter, that is, from the requirement that all individual voters be treated equally. It is thus very different from the usual arguments that proportional representation (PR) is "fair." Frequently the fairness of proportionality is simply assumed. For example, Lijphart (1994, 140) concludes that for many PR supporters, proportionality is simply a goal in itself, "virtually synonymous with political justice." When proportionality is given justification, this tends to be in terms of fairness to political parties or social groups, or in terms of the desirable instrumental effects of proportional representation. For example, McLean (1991) argues that the case for proportional representation rests on the idea that a legislature should be a microcosm of the population, as did Black (1958/1971). Similarly, Pitkin (1967) identifies the case for PR in terms of "descriptive representation" (the fair representation of every salient group), while Still (1981) uses the similar concept of "group representation." The political electoral reform discourse frequently emphasizes the unfairness of nonproportional representation to certain political parties, who win far fewer seats than their vote share would entitle them to under proportionality. This argument can be rephrased in individual terms as the demand that the same number of voters for each party be needed to elect a representative (for example, Beetham 1992; Jenkins 1998). Of course, arguments for PR in terms of fairness to political parties or social groups may be convincing to a considerable number of people. However, the point here is that they are not the only arguments, nor I think the most fundamental.

Arguments against proportional representation are often based on the idea that some other value (such as stability, economic performance, or accountability) outweighs the fairness of PR (to be considered in chapters 6 and 8). However, it is sometimes argued that the case for proportional representation rests on one particular kind of fairness or equality (numerical fairness to parties or social groups), and that other forms of fairness (e.g., the winner-take-all principle or the constituency principle) would lead to different conclusions (see, for example, Beitz 1989 and various contributors to Jenkins 1998). This argument is a direct result of thinking about equality and fairness in group terms. If we consider equality or fairness in terms of groups, it is possible to divide society into groups in many ways, and it is possible to find various principles to arbitrate between their claims. For example, if we consider equality between social groups, we get descriptive representation and a justification for PR; whereas if we consider equality between geographical areas, we get the constituency principle and an argument for single-member district systems.

It should be noted that the argument made in favor of the fairness of first-past-the-post elections rests on conceptions of fairness to groups or political parties every bit as much as the usual arguments in favor of PR. For example, the winner-take-all principle considers fairness in terms of parties or candidates, not voters. First-past-the-post is fair, it is argued, because the party that wins a fair contest gets the prize. Similarly, the constituency principle is based on fairness to geographically defined groups of people. Of course, advocates of these principles prefer to phrase them in individualist terms (the rights of individual residents of a constituency, or of voters of the plurality party), just as advocates of PR do. However, as with the usual argument for PR, these arguments are still group-based in that they conceive individual voters in terms of a preconceived group identity, instead of conceiving them as individuals per se in the liberal sense.

By basing the argument on a liberal conception of equality—that is, the idea that all individual voters should be treated equally—we avoid this relativism. We no longer have to decide which groups deserve special consideration but merely have to make the system fair to individuals. Individuals can then decide which group identities are salient to them when they vote. As a result, it is possible to come to a determinate conclusion—political equality implies proportional representation. The previous chapter has justified why we should prefer the liberal conception of equality over group-based conceptions—it can be justified either in terms of doctrinal liberalism (the rights of individuals have precedence over the rights of groups those individuals make up) or in terms of individual equality providing the only democratic means to arbitrate between the claims of various cross-cutting groups.

Whereas our analysis starts with basic normative principles, such as political equality, and sees what this logically requires in an electoral system, the empirical electoral system literature starts with existing electoral systems and studies their effects. Much of the electoral system literature has focused on the effect of electoral rules on party systems. Duverger (1954/1963) found that first-past-the-post elections tended to produce two-party systems, while proportional representation produced multipartism. Rae (1967) systematically compared district magnitude (the number of candidates elected from each district) and electoral rules to explain cross-national differences in proportionality, large-party advantage, and the number of parties. More recent works in this tradition include Taagepera and Shugart (1989) and Lijphart (1994).

When it deals with normative questions of democracy, the electoral systems literature tends to operate in instrumental terms, as typified by the title of Powell's (2000) book *Elections as Instruments of Democracy.* Various conceptions of democracy are set out, and different electoral systems are evaluated in terms of how far they produce results compatible with these conceptions. Thus in Powell's account majoritarian conceptions of democracy stress the direct accountability of government to the electorate, as operationalized by how likely a change in popular support is to produce a change in government; while proportional conceptions of democracy see democracy as a multistage process requiring "authorized representation," measured in terms of what proportion of the voters voted for a government party, and the degree to which policy outcomes match the preferences of the median voter. Plurality systems do well on the first set of criteria, while proportional systems do well on the second. Similarly Lijphart (1994) contrasts the value of proportionality maximized by proportional systems with the accountability provided by plurality elections. Katz (1997) goes even further, providing a long list of conceptions of democracy (including less credible variants such as "guided democracy," socialist "people's democracy," and Calhounian veto-group "democracy") and tracing the type of election systems required by each. For the most part there is a studied impartiality between plurality and proportional election systems, although there are exceptions. (Lijphart [1999] links proportional election systems with favorable outcomes in terms of factors such as economic equality, quality of life, and environmental protection, while providing similar outcomes in terms of economic growth and stability. Dummett [1997], while being very critical of first-past-the-post and single transferable vote, allows that the choice of replacement depends on competing principles, although he does propose a new system based on a modified Borda procedure. Farrell [2001], while accepting that there is a trade-off between the accountability provided by plurality systems and the accurate representation provided by PR, argues that PR is preferable because the main argument against PR—that it produces unstable government—is empirically untrue.)

The social choice literature studies the axiomatic properties of voting procedures. It thus provides a means for taking values from the normative literature and translating them into rigorous requirements that can be applied to empirical electoral systems. However, the social choice literature has concentrated on social decision rules (where a decision is to be made among competing alternatives) as opposed to seat allocation rules (where political representation is distributed). Most previous technical work on proportional representation has tended to concentrate on the mechanics of seat allocation rules rather than the axiomatic justification of the principle of proportionality (see, for example, Balinski and Young 1982/2001; Taagepera and Shugart 1989).

There is, however, some axiomatic work dealing with the desiderata of electoral systems. Dodgson (1884/1995) advocates a PR system primarily on grounds of individual fairness. The first two desiderata of an electoral system he gives are that every voter has the same chance of being represented and that every represented voter be represented by the same fraction of a member. Ward (1995) uses computer simulations of one-dimensional party competition to argue that proportional representation is the electoral system most likely to produce policy choices close to those of the median voter. Feld and Grofman (1986) consider how a representative system can replicate the preferences of a population as a whole, while Benoît and Kornhauser (1994) show that distributed representation can lead to Pareto-inferior results. Monroe (1995) and Chamberlin and Courant (1983) propose new electoral systems based on maximizing representation. Deemen (1993) shows that certain voting paradoxes apply to list PR systems. Rogowski (1981) suggests that anonymity implies proportionality, but he provides no proof. Hout, Swart, and Veer (2002) show that anonymity, neutrality, consistency, faithfulness, and topsonlyness imply the plurality ranking property; the results presented here draw on this insight. Hout and McGann (2004) show that anonymity, neutrality, and positive responsiveness imply a result equivalent to pure list proportional representation.

This chapter is divided into two sections. The first section sets out the main theoretical result, that political equality implies proportionality in single-vote electoral systems. It also shows that this result can be extended to multiple-vote electoral systems, in that political equality requires that these systems produce proportional results if the preferences of voters correspond to electoral lists. Although the exposition is intuitive (formal proofs are reserved for the appendix) this section is somewhat technical. The second section considers electoral systems in practice. After laying out a typology of electoral systems, it considers how well each system conforms to the ideal of proportionality. It then briefly lays out other considerations that may be significant when comparing electoral systems. This discussion of other effects of electoral systems is continued in parts 2 and 3 of the book, particularly in chapter 6.

1. Theory: Political Equality Implies Proportionality

Political equality logically entails proportional representation, based on the proofs in Hout and McGann (2004). This section relies on the liberal conception of political equality defined in the last chapter. If we consider political equality in terms of individual voters, then equality and impartiality become equivalent. Thus we can proceed negatively, defining equality in a seat allocation rule as not being biased—that is, not taking into account inappropriate considerations. In this way we can axiomatize political equality as the qualities of anonymity and neutrality. That is to say, a seat allocation rule satisfies political equality if it does not discriminate between voters on the basis of their identities (it is *anonymous*) and does not discriminate between alternatives (candidates, lists, or parties) on the basis of their identities (it is *neutral*). As argued in the previous chapter, it is necessary to apply anonymity and neutrality not just to individual alternatives but also to coalitions of alternatives, as decisions in a legislature will sometimes depend on the relative size of coalitions. Formal axiomatizations are given in the appendix.

We also need to define what is meant by *proportional representation.* This may seem unnecessary. However, we will see that the concept of proportional representation and its implementation differ significantly. In particular, the concept of proportionality does not require that we think in terms of fairness to political parties. Indeed, given that our conception of liberal political equality is based on the equality of individuals, it is important that we be able to define proportional representation in a way that does not depend on parties.

The Concept of Proportional Representation

We define pure proportional representation as a seat allocation rule that assigns seat share to alternatives in proportion to their vote totals (as before, alternatives can be candidates, lists, or parties). The concept of pure PR is of course an abstraction. No existent electoral system meets this criterion, although some come close. It would be theoretically possible to allocate legislative weight to alternatives in direct proportion to vote share in the manner suggested by Chamberlin and Courant (1983). However, if we insist that the votes of all legislators have equal weight, the fact that seats are not infinitely divisible means that there is always some divergence from proportionality, although in principle this could be overcome.[1] Other features of many existing PR systems, such as thresholds and small districts, also reduce proportionality. Nevertheless, although there are no empirical examples of pure PR, it serves a purpose here as a counterfactual ideal, and there are existing systems that approximate it quite well.

It should be noted that although it is often assumed that proportional representation must be based on political parties and party lists, this is not the case. Some existing PR systems are based on lists of citizen candidates (who may or may not be affiliated with a party), and it is even possible to define a PR system in terms of individual candidates. Dodgson (1884/1995) proposes such a system. Each voter casts one vote in a large electoral district. Every candidate who received a quota, that is, total votes cast / (number of seats + 1), is elected. Candidates are then able to distribute any surplus votes they may have received in any way they wish. The intuition is that the votes a candidate receives are their property to dispose of at will. Dodgson defends this in terms of the concept of representation. If I am willing to choose someone to make legislative decisions on my behalf, I surely trust this person enough to choose their own deputies to carry out this legislative task.

List proportional representation can be seen as a set of restrictions on the system proposed by Dodgson. Suppose that we require candidates to state in advance how they would distribute their surplus votes. Then we have a rudimentary system of list PR. Of course, all existing list PR systems place considerably more constraints on candidates. Generally, candidates are only allowed to have their names on one list, and all candidates on the list are required to distribute their excess votes in the same way defined by the list. In practice most systems of list PR force candidates to compete as mutually exclusive teams.

This, however, does not force candidates to be organized as political parties. A party is not just a slate of candidates but rather a sociological organization that combines a slate of candidates with an organizational structure, usually a parliamentary fraction, and sometimes affiliated social organizations. Some list PR systems, such as that in Germany, require that lists be affiliated with registered political parties, and thus can be referred to as party-list PR systems. However, other systems, such as that in the Netherlands, afford parties very little in the way of special treatment. Rather, elections are organized in terms of lists of candidates, which any group of citizens can propose. It is true that most lists are affiliated with political parties. However, it is also true that virtually all members of Parliament under the plurality system in the United Kingdom are also affiliated with one of the main parties. Modern politics creates strong incentives for candidates to join parties (see Aldrich 1995), regardless of the electoral system. What is important from a liberal point of view is that while the list PR systems in countries such as the Netherlands accommodate political parties, they do not mandate them. Indeed, hastily assembled citizen lists have won considerable vote share in some recent elections in the Netherlands.

Results

Political equality (operationalized as the axioms of anonymity and neutrality) implies a single-vote seat allocation rule essentially equivalent to pure proportional representation. First, it is shown that any single-vote seat share allocation rule that is positively responsive, neutral, and anonymous satisfies the strong plurality ranking property (alternatives that win more votes get more seats). This result applies to coalitions as well as alternatives (if the seat allocation rule is anonymous and neutral, coalitions whose members win more votes must get more seats in aggregate). In parliaments, governments are typically chosen by majority rule, the vote of investiture usually requiring a coalition. Therefore the outcome depends on the coalition formation game defined by the election and the seat share allocation rule. It is shown that any seat share allocation rule that is anonymous and neutral (and thus satisfies the strong plurality ranking property) defines a coalition game identical to that defined by pure list proportional representation. We show similar results when the seat share allocation rule is assumed to be nonnegatively instead of positively responsive.

First it is necessary to show that anonymity, neutrality, and positive responsiveness imply the strong plurality ranking property—if alternative A wins more votes than alternative B, then it must receive a greater seat share. If we only require nonnegative responsiveness, then anonymity and neutrality imply the weak plurality ranking property (alternatives that get more votes must win at least the same number of seats).

PROPOSITION 1: *Any seat share allocation rule that is anonymous, neutral, and positively (nonnegatively) responsive satisfies the strong (weak) plurality ranking property.* (Proof in appendix.)

The intuition here is straightforward. If two alternatives have the same number of votes, then by anonymity and neutrality, they must have the same seat share. If in this case one alternative receives more seats, then either the vote allocation system is inherently biased in its favor (violating neutrality) or some voters' votes count for more than others (violating anonymity). If one alternative then increases its vote at the expense of the other or by gaining the votes of people who previously abstained, by positive responsiveness it must receive a greater seat share than the other alternative. Therefore, if one alternative receives more votes than another it must receive more seats (the strong plurality ranking property). If we only assume nonnegative responsiveness, then when an alternative gains votes, it must at least not lose seats. This implies that an alternative that wins more votes than another must get at least an equal seat share (the weak plurality ranking property).

Proposition 1 can be extended to apply not only to seat allocation for individual alternatives, but to seat allocation for coalitions of alternatives. Strictly speaking we define a seat share allocation rule as anonymous, neutral, and positively responsive to coalitions if the allocation of seats to coalitions also satisfies anonymity, neutrality, and positive responsiveness, however we partition the alternatives into coalitions.

PROPOSITION 2: *Any seat share allocation rule that is anonymous, neutral, and positively (nonnegatively) responsive for coalitions satisfies the strong (weak) plurality ranking property for coalitions.* (Proof in appendix.)

The proof of this proposition is essentially identical to that of Proposition 1. If one coalition of alternatives receives more votes than another, it must receive a greater total seat share if we assume positive responsiveness, and at least an equal total seat share if we only assume nonnegative responsiveness.

In representative bodies, governments are typically formed by a process of majority-rule coalition formation. Which coalition forms is a result of a bargaining process. However, the bargaining situation is defined in terms of which coalitions have sufficient seats to win a majority-rule vote of investiture (or confidence) and form a government. It can be shown that any seat allocation function that satisfies the coalitional strong plurality ranking property defines a coalition formation game identical to that defined by pure PR. Therefore anonymity, neutrality, and positive responsiveness imply a single-vote seat share allocation that produces a parliamentary outcome to all intents and purposes identical to that produced by pure PR.

> PROPOSITION 3: *Any seat share allocation function that is anonymous, neutral, and positively (nonnegatively) responsive for coalitions defines a majority rule coalition game with a set of winning coalitions that is identical to (a subset of) that defined by seat share allocation by pure proportional representation.* (Proof in appendix.)

The intuition behind the proof comes from the fact that under majority rule a coalition is winning if it has a greater seat share than all the alternatives excluded from it. By Proposition 2, anonymity, neutrality, and positive responsiveness require that if a coalition has more votes than another coalition, it must receive a greater seat share. Therefore the set of winning coalitions must be the set of coalitions whose members have more votes than all the alternatives excluded by them. This is exactly the same as the set of winning coalitions under pure PR.

If we only assume nonnegative responsiveness, it is possible for a coalition to win a majority of the votes but only to receive exactly half the seats, and thus be a blocking but not winning coalition. However, it is impossible to have a "manufactured majority" (a situation where an alternative or coalition with a minority of the vote gets a majority of the seats) without violating anonymity or neutrality. The intuition behind this is similar to the case assuming positive responsiveness. By Proposition 2, anonymity, neutrality, and nonnegative responsiveness require that if a coalition has more votes than another coalition, it must receive at least an equal seat share. It is possible for a coalition to have an equal number of seats to its complement, so it is possible for a coalition to win a majority of the vote, but to only receive exactly half the seats. Any winning coalition under our seat share allocation rule is a winning coalition under proportionality but not vice versa.

Extension to Multiple-Vote Systems

The formal results apply to single-vote electoral systems. However, the principle of proportionality is still relevant when considering electoral systems that ask voters for several choices. We can classify multiple-vote electoral systems into two groups. First there are so-called mixed-member systems (Shugart and Wattenberg 2001). These typically give voters a vote for a candidate to represent their district and a vote for a party, but only ask for the voter's first choice in each category. We will argue that these systems can easily be accommodated within our framework. Second there are ordinal rules that ask voters to rank alternatives, such as single transferable vote (STV) and the Borda procedure. It is more difficult to define what we mean by proportionality when voters order candidates instead of casting a single vote. However, in the special case where voter preferences truly correspond to electoral lists, then liberal equality still implies proportionality, as per our result. Thus, to satisfy the condition of political equality a multivote electoral system has to be compatible with proportionality, if all voters choose to vote for a straight list.

Mixed-member systems can be understood as a combination of two single-vote seat share allocation rules. These systems can be divided into two groups—mixed-member plurality and mixed-member proportional (Shugart andWattenberg 2001). An ideal-type mixed-member plurality system typically allocates a certain number of seats to district elections and a certain number to proportional election, with no compensation between the two. The proportional part of the election largely respects the principles of anonymity and neutrality, while the district election typically does not. Consequently, the overall result will violate the principles of political equality we have defined. An ideal-type mixed-member proportional system, on the other hand, distributes seats from the proportional part of the election in a compensatory manner, so that overall seat totals of each party from both stages approximate proportionality. As a result, these systems essentially function as list PR and thus approximately respect political equality.[2]

We can show that ordinal voting systems that respect political equality must be compatible with proportionality when voters have list preferences. Voters have list preferences if they all rank the candidates on one electoral list in the order of the list, and are indifferent between all the candidates not on that list. If this is the case, the preferences of every voter can be summarized by a voting correspondence (voter x supports list y). If this is so, we can apply the theorems of the last subsection and show that liberal equality implies proportionality in this case. Of course, the fact that there is an ordinal voting system gives the voters many options other than voting a straight list. However, if they choose to vote in this way, liberal equality implies that the result should be proportional. The idea of list voting here is intended as a thought experiment. However, it may not be that far from reality in many countries. For example, in the Netherlands over 90 percent of voters routinely vote for the national leader of their preferred party, in spite of having the option of voting for anyone on the party list (see Gladdish 1991).

It should be noted that the normative case for considering voters' entire preference schedule is questionable. It has frequently been stated as obvious that a good election rule should do this (Black 1958/1971, 95; Dummett 1997). However, we can make a normative case for considering only the first preference of voters in allocating seats in legislatures. Under a pure proportional system (an abstraction, of course, given that seats are not divisible in reality), everyone gets a representative of the list they choose, however small. In a sense there is no need to consider second-place preferences because everyone gets their first preference (see Dodgson 1884/1995 for an early statement of this position). Furthermore, it can be argued that the fact that my preferred representative is your very least preferred is irrelevant—their job is to represent me, not you. While this argument is plausible, it requires a stronger theory of representation, such as Powell's (2000) concept of authorized representation. The necessary assumptions are far more demanding than the minimum of liberal equality required in the last section.

We can discuss the use of the most commonly studied ordinal vote mechanisms as seat allocation rules. Both single transferable vote[3] and the Borda count[4] are anonymous and neutral in terms of individual candidates but only if applied to a single district. If voters have list preferences, single transferable vote produces results compatible with proportionality. However, in practice single transferable vote is usually applied to many, rather small districts (in Ireland, the size varies between 3 and 5), rather than a single national district. For this reason, Farrell (2001) finds that single transferable vote in Ireland does not produce strict proportionality, but that it is far closer to it than plurality elections. It is possible to apply single transferable vote to national elections, but this would result in ballots with hundreds, if not thousands of candidates. One solution to the resulting problem of unwieldiness would be to allow voters to vote a straight party ticket. However, if most voters act in this way, the results will be virtually identical to list PR.

The Borda count in general does not satisfy proportionality if voters have list preference, and it has some other features that make it extremely problematic as a seat allocation rule. This is not surprising, as it was originally proposed as a rule for ranking candidates, not for distributing

Voter	1	2	3	4
	a	a	b	b
	b	b	a	a

Fig. 3.1. Configuration of voters under Borda count

Voter	1	2	3	4
	a	a′	b	b
	a′	a	a′	a
	b	b	a	a′

Fig. 3.2. Configuration of voters under Borda count with new party

representatives. Suppose we have two parties a and b, each of which has two voters who favor it, as shown in figure 3.1. Parties a and b both receive a Borda count of 2 and thus receive an equal allocation of seats.

However, now let us assume that a faction breaks away from party a to form party a′, giving us the preference distribution shown in figure 3.2. Party a, party a′, and party b now all get a Borda count of 4, so all get equal representation. However, this means that the combined representation of parties a′ and a is now double that of b. By dividing in two, the original party a has increased its representation at the expense of b. This property of the Borda procedure makes sense when we are ranking candidates—if a new candidate enters the race who is almost identical to a, that candidate should score almost identically to a. However, it is not a desirable quality when distributing seats, because it does not take into account the similarity of candidates or parties. This leads to some potentially undesirable consequences, such as encouraging party fragmentation and possibly excluding minority representation. Apart from these consequences, the results will be arbitrary as they depend as much on the number of candidates of each type running as on the preferences of the voters.

To mitigate these problems, Dummett (1997) suggests a hybrid "quota Borda system" that combines a Borda procedure with a provision that candidates who receive a certain quota of first-place votes are automatically elected. This is essentially a combination of single nontransferable vote with the Borda count, with single nontransferable vote electing candidates receiving a quota of first-place votes and Borda electing the rest. As such it is likely to inherit the problems of SNTV, such as a high premium placed on a party's supporters distributing their votes between candidates optimally (see Cox 1997; Bowler and Grofman 2000). In addition, it does not address the problem of the results being arbitrary in that they are dependent on the number of each type of candidate running.

Other seat allocation rules have been proposed that are technically variations of the Borda count but have quite different effects and goals. Chamberlin and Courant (1983) propose a system based on a Borda-type procedure to maximize the "representativeness" of a committee, in the sense of maximizing the number of people who have a highly ranked candidate on the committee. This would lead to candidates with relatively few votes being overrepresented, so to compensate for this, voting in the committee would be weighted by the number of votes received, so the voting strength in the committee would be identical to ordinary proportional representation. Monroe (1995) proposes a generalization of proportional representation that he refers to as "fully proportional representation." The ordinal implementation of this is technically related to the Borda count. The procedure considers every possible partition of the voters into equally sized groups and assigns each group its Borda winner as its representative. The partition for which the elected representatives best fit the groups is then selected. Monroe claims that the procedure is not practical due to the ease with which it could be manipulated by strategic voting. However, the procedure may prove to be theoretically important in that it may provide a limiting case for proportional representation with candidates rather than lists (the Chamberlin and Courant 1983 procedure may have a similar utility).

So, although our technical results only refer to single-vote seat allocation rules, they can be applied to multiple-vote rules. If voters have list preferences, then liberal equality still implies that the seat allocation rule be proportional. Essentially, political equality requires that an ordinal seat allocation rule be compatible with proportionality, if voters vote in terms of lists rather than candidates. Some ordinal voting systems are compatible with this kind of proportionality, such as single transferable vote and the Borda-type procedures proposed by Chamberlin and Courant (1983) and Monroe (1995), although the Borda count itself is not. It should be noted that it is far from obvious that rules that consider a voter's entire preference profile are normatively superior to rules that only consider first preferences.

2. Electoral Systems in Practice

The previous section showed that the principle of political equality implies proportionality in single-vote electoral systems. It also showed that multiple-vote electoral systems that respect political equality either produce results very similar to proportional systems or are problematic for other reasons. However, proportionality is only a principle: We need to consider how this principle can be translated into institutional practice, given that no existing electoral system is perfectly proportional. This section proceeds by first laying out a typology of electoral systems. It then considers how well they approximate the ideal of proportionality. Finally, other effects of electoral systems are considered.

Typology of Electoral Systems

There is a well-developed literature classifying electoral systems that we can draw upon (see, among others, Rae 1967; Taagepera and Shugart 1989; Lijphart 1994; Katz 1997; Farrell 2001). We can begin by classifying pure electoral systems (systems that use one procedure to assign seats, as opposed to mixed electoral systems that use a mixture of procedures). Following Rae (1967) we can classify these along two dimensions, district magnitude and electoral formula.[5] District magnitude is simply the number of seats distributed in each electoral district. This can range from 1 in a country like the United States with single-member district elections to 150 in the Netherlands, which has a single nationwide district.[6] The electoral formula is the rule used to allocate seats (plurality, proportional representation, single transferable vote, etc.). Table 3.1 summarizes the various combinations.

The simplest formula is plurality, where the candidates are simply ranked according to how many votes they receive. If there is only one seat to be distributed, this gives us single-member district plurality (first-past-the-post) elections where the highest vote-getter is elected, regardless of whether that candidate receives an absolute majority of the vote. A variant of plurality in a single-member district is plurality runoff, sometimes (mistakenly) called "majority-rule runoff."[7] Under this rule, the candidates are ranked by plurality, and all but the top two candidates are eliminated. Another vote is then taken to determine the winner. If

TABLE 3.1. Typology of Pure Electoral Systems

	District Magnitude	
Formula	**Single Member**	**Multimember**
Plurality	Single-member district plurality (first-past-the-post) Plurality runoff	Single nontransferable vote Multiple vote
Proportional	—	List proportional representation
Ordinal vote	Single transferable vote Borda	Single transferable vote Borda

we apply plurality to multimember districts, we get single nontransferable vote or multiple vote, depending on how many votes each voter gets. If there are three seats to be distributed, under single nontransferable vote, each voter gets 1 vote, and the three candidates with the most votes are elected. Multiple vote works the same way, with the exception that each voter may get 2 or 3 votes.[8]

Under list proportional representation a list of candidates receives seats in proportion to the number of votes it receives, so if a list wins three seats, the first three names on its list are elected. (In some systems the lists have to be associated with political parties, but this is not always the case.) This only makes sense with multimember districts. However, various ordinal formulas, such as single transferable vote and the Borda count, make sense with either single- or multimember districts (see section 1 for definition of these ordinal rules).

Proportional representation actually refers to a variety of formulas that approximate proportionality. No formula can be exactly proportional unless it divides seats. For example, if we have a 5-seat district and a list wins 30 percent of the vote, it can either win 1 seat (20 percent of the total) or 2 seats (40 percent), but not 1.5 seats. This is less of a problem if we have a large district magnitude, as it is possible to get very close to proportionality. With relatively small district magnitudes, however, the choice of formula can have a considerable effect. There are two families of formulas, quota and divisor.[9] With quota methods, first the quota needed to elect a candidate is determined. The list with the most votes is awarded a seat. A quota's worth of votes is then subtracted from its total. This is repeated until all seats are allocated. Different quotas may be used, as summarized in table 3.2. Generally the Hare quota gives less of an advantage to the list with most votes than the Droop quota, which in turn has less large-list bias than the Imperiali quota (Taagepera and Shugart 1989). With divisor methods, there is a series of divisors. The largest list is awarded a seat, and then its vote is divided by the first divisor. Each time a list wins a seat, its vote is divided by the next divisor.

TABLE 3.2. Proportional Representation Formulas

Type of mechanism		
Quota	Hare	Quota = voters/seats
	Droop	Quota = (voters/(seats + 1)) + 1
	Imperiali	Quota = voters/(seats + 2)
Divisor	D'Hondt	Divisors: 1, 2, 3, 4, . . .
	Sainte-Laguë	Divisors: 1, 3, 5, 7, . . .
	Modified Sainte-Laguë	Divisors: 1.4, 3, 5, 7, . . .

This is repeated until all seats are filled. Of the divisor methods, D'Hondt has the greatest large-list bias, followed by modified Sainte-Laguë and Sainte-Laguë (Taagepera and Shugart 1989).

Two other features of proportional representation mechanisms are particularly notable: whether the list is open or closed, and whether there is a minimum threshold required for a list to win any seats. With closed-list proportional representation, the list determines the order in which candidates are elected. With open-list PR the voters are able to determine the order of candidates on the list. For example, in Italy prior to 1994 voters had two votes, one for the party and one for their preferred candidate. The party vote determined how many seats the party received, while the individual preference vote determined who filled those seats. There are also many systems that can be described as semiclosed, in that it is theoretically possible for voters to change the list order but very difficult in practice. The Netherlands is an example of this.[10] Many proportional representation systems have electoral thresholds, so that if a list does not exceed this threshold it receives no seats, even though it would be entitled to some seats by proportionality. For example, in Turkey a party that wins 9.9 percent of the vote receives no seats, whereas it would receive 55 seats if it made the 10 percent threshold.

In addition to pure electoral systems, where one mechanism is used to distribute seats, there are mixed-member systems, where different mechanisms are used to distribute different seats. Typically the election is divided into local and either regional or national seats (or in some cases both). In nearly all cases the national seats are distributed by some form of proportional representation. According to Shugart and Wattenberg (2001), the key variable is whether the national seats are compensatory or not. If the national seats are compensatory, then they are distributed so as to restore proportionality to the overall result. Thus if a party wins a disproportionate number of local tier seats, it receives fewer of the national seats to restore proportionality. However, if the system is not compensatory, the party is allowed to keep all the local tier seats it won, and it wins a number of upper tier seats proportionate to its vote. A second variable is whether the lower tier seats are distributed from single-member districts or multimember constituencies. The various combinations are summarized in table 3.3. If we have single-member district elections to the lower tier and compensatory proportional representation at the upper tier, then we have a mixed-member proportional system, to use Shugart and Wattenberg's (2001) term. Germany is the most noted example of this. Each voter has two votes, one for a single-member district candidate and the other for a party. Seats at the regional (Land) level are distributed so that the overall number of seats each party receives is proportional to the number of party votes. If we have single-member district elections for the lower tier, and noncompensatory proportional representation for the upper tier, we have mixed-member plurality (or what Shugart and Wattenberg refer to as mixed-member majoritarian).[11] An example of this would be the Italian lower house where three-quarters of the seats are chosen by single-member district elections and one-quarter are chosen by proportional representation.

It should be noted that many existing proportional representation systems are actually multiple-tier systems, something that Lijphart (1994) particularly emphasizes. List proportional representation is used for the lowest tier. However, because the districts at this level are quite small, considerable disproportionality can result. To counter this there are regional and/or national seats that are allocated to correct this. Typically such systems give the voter only a single vote, so their local tier vote is tied to their national vote, although it would be theoretically possible to implement such a system with a separate vote for each tier. Countries with such a system include Austria, Belgium, Denmark, Norway, and Sweden. It would also be possible to use other multiple-member district systems for the lower tier, such as single nontransferable vote. Another possibility would be to combine list proportional representation at the lower and upper tiers in a noncompensatory manner, although it is not clear what would be achieved by this.

Thus we have a typology of existing electoral systems. While most of the variation can be captured in the simple dichotomy between plurality and proportional representation, we will see that the institutional details, particularly between different implementations of proportional representation, have considerable effects.

TABLE 3.3. Typology of Mixed-Member Systems

	District Magnitude, Lower Level	
Mechanism	**Single Member**	**Multimember**
Compensatory	Mixed-member proportional	Multitier proportional representation
Not compensatory	Mixed-member plurality (Mixed-member majoritarian in Shugart and Wattenberg 2001)	— (but theoretically possible)

Proportionality in Practice

It is common knowledge that proportional representation produces results that are approximately proportional, while plurality (whether single-member district, runoff, or single nontransferable vote) can produce considerable large-party bias. However, the details are considerably more complex. For example, small-district proportional representation (for example, as in Spain) can also produce a considerable large-party advantage, while national district PR in the Netherlands produces results as close to proportionality as is possible without dividing seats. In India, single-member district plurality elections produce results that are approximately proportional, because there are many regionally concentrated parties. Furthermore, if the goal is to produce proportionality, this can be done with many mixed-member systems, as well as with pure proportional representation.

First it is necessary to define what we mean by *proportionality.* The quality of proportionality used in the theoretical portion of this chapter applies to the electoral mechanism, not to the outcome. An electoral system is proportional if it translates *x* percent of the vote into *x* percent of the seats for *any* party, real or hypothetical. This is not the same as the definition used in much of the empirical literature. Instead this literature considers how proportional the results are, comparing the number of seats won by actual parties with the share of the votes won. This relationship is either plotted to form a proportionality profile (Taagepera and Shugart 1989) or the differences are combined to produce a single measure of disproportionality (see Lijphart 1994; Taagepera and Shugart 1989 for discussions of the various indices). It is possible for an extremely disproportional system to produce proportional results. For example, in the United States, plurality elections to the House of Representatives typically produce results that are quite close to proportionality, because parties that would be severely underrepresented either do not run or people do not vote for them. Similarly, in India regional concentration produces approximately proportional results. However, we would not consider the electoral system in India proportional, because a small to medium-sized party with a geographically dispersed following would be severely underrepresented. Nevertheless, providing we proceed with caution, we can draw on the empirical electoral systems literature to discuss the relative proportionality of various electoral rules.

The two main variables to be considered are the electoral formula and the district magnitude, as in the previous section. In terms of the electoral formula, there is obviously a large difference in proportionality between proportional representation and plurality rules. However, which proportional representation rule is used makes little difference (Katz 1997). This is because all the proportional representation formulas tend toward proportional outcomes as the district magnitude gets large. It is only with small district magnitudes that the difference between the various rules makes a difference. Lijphart (1994) finds that the various proportional formulas can be ranked in order of proportionality as follows: (1) Hare, Sainte-Laguë; (2) Droop, Modified Sainte-Laguë; (3) D'Hondt, Imperiali. Single transferable vote is hard to classify as it applies to individual candidates, not parties. However, assuming that people vote party line, Lijphart (1994) finds that single transferable vote is roughly as proportional as Droop and modified Sainte-Laguë. This contrasts to some of the empirical literature, which suggests that STV is not very proportional (see Farrell 2001). This, however, is largely due to the fact that STV is typically implemented with small district magnitudes (3–5 in Ireland), which reduces proportionality.

District magnitude has a strong effect on proportionality, as has been noted since Rae (1967). Indeed Taagepera and Shugart (1989) suggest it is the decisive factor in determining proportionality in that it explains most of the variance in proportionality and other outcomes, such as the number of parties and fractionalization of the party system, especially between countries with PR formulas. Katz (1997), however, finds that higher district magnitude does not lead to more proportionality under a plurality formula (that is, as we move from single-member district to single nontransferable vote with higher magnitudes), although the number of cases is small. Electoral thresholds clearly depress proportionality, as they lead to parties below the threshold being unrepresented, with other parties overrepresented as a result. Taagepera and Shugart argue that it is possible to combine the effect of district magnitude with that of thresholds to produce a single measure (adjusted district magnitude) that characterizes the effect of the electoral system.

Mixed-member systems can produce highly proportional results providing that they are compensatory and there are enough upper tier seats to compensate for any disproportionality resulting from the lower tier allocation. Noncompensatory mixed-member systems will preserve whatever disproportionality exists in the lower tier, and thus can produce results that vary considerably from proportionality, unless most of the seats are in the proportional upper tier, as in the Polish lower chamber. Multitier PR systems are likely to be extremely proportional, as the lower tier allocation by small district PR typically produces only slight deviations from proportionality that can be easily corrected by the upper tier, even if the number of upper tier seats is small.

Given that proportional formulas and large district magnitudes produce proportional outcomes, if the goal is to ensure proportionality, the most obvious electoral system is national list PR. However, there are other systems (mixed-member, multitier PR) that can produce results almost as proportional. Thus, to choose between competing electoral systems that satisfy proportionality, it is necessary to consider other factors. Although much of this discussion will take place in later chapters, the next section briefly summarizes some of these considerations. I will argue that a strong case can be made for national list PR against mixed systems, although the conclusion will be tentative and rest on subjective value weightings.

Other Effects of Electoral Systems

Although there are many procedures that can satisfy proportionality other than national list PR, these systems produce results that are quite different in other respects. Later chapters will consider these arguments in more depth (particularly section 3 of chapter 6, on representation). Nevertheless we can summarize these considerations. National list PR (or failing that, large district magnitude PR) gives the strongest incentives for representatives to pursue a broadly defined public good, while small district magnitudes provide incentives for particularism and even pork barrel politics. Of course, the flip side of this is that small districts give more incentive for representatives to actively pursue local interests. Furthermore, large districts increase the probability that representation will be descriptively accurate, in terms of categories such as gender and ethnicity. It is frequently argued that plurality elections provide more accountability and responsiveness to shifts in public opinion. In chapter 6 we show that this claim is fundamentally flawed. Finally, district magnitude affects the number of parties and the ease with which new parties can enter the system. Larger district magnitude leads to higher party system fractionalization and a higher effective number of parties (Rae 1967; Taagepera and Shugart 1989).[12] A larger district magnitude also means that the vote that a party needs to gain representation is lower, providing there is not a legal threshold. Thus large effective magnitude systems are more competitive than systems with low effective magnitude, in the sense that there are fewer barriers to entry for new parties.

Summary

This chapter has shown that the basic value of political equality implies that an electoral system must be proportional. More precisely, we have shown that any single-vote seat share allocation rule that satisfies anonymity, neutrality, and positive responsiveness must produce results that are to all intents and purposes identical to those produced by proportionality. We obtain similar results assuming nonnegative instead of positive responsiveness. Furthermore, political equality implies that multiple-vote systems must be compatible with proportionality, if voters' preferences correspond to electoral lists. This has notable consequences for the theory of democracy. Much of the previous empirical literature on electoral systems has been extremely agnostic as to ends, the argument being stated in the form "If you desire *x,* then choose electoral system *y.*" Similarly, Dahl (1956) argues that the axiomatic approach has little to say about the practice of democracy because it does not deal with representative systems. In contrast, the results of this chapter show that the basic axiom of political equality places strict requirements on electoral systems, namely, that they satisfy proportionality.

Of course, there are many electoral systems that can satisfy proportionality. The most obvious is, of course, list PR. However, mixed-member systems with compensatory seats and multitier PR will also produce proportional results, as will single transferable vote and the Borda-type systems devised by Chamberlin and Courant (1983) and Monroe (1995), provided they are applied to national districts. To choose between these different systems requires other considerations be taken into account, such as the type of representation, accountability, and deliberation desired. This will be considered in chapter 6. For now let us note that the value of political equality implies proportionality, which eliminates many existing electoral systems, including some of the less proportional forms of PR. If such systems are to be justified, a case has to be made in terms of other values that on balance are argued to be more important than political equality. Parts 2 and 3 of this book will consider other such values.

APPENDIX: PROOFS

All proofs in this chapter are based on Hout and McGann (2004).

Let us define the set of eligible voters as N, with voters numbered $1 \ldots n$, and the set of alternatives A, numbered $1 \ldots a$. The voting correspondence V is defined over the Cartesian product $N \times A$, with ${}_{i \in N}V_{j \in A}$ reading "i votes for alternative j." Assigning the value 1 for true and 0 for false, $\forall\ i \in N \sum_{j \in A} {}_iV_j \leq 1$. (Each individual either votes for one alternative or does not vote.) The function T maps the voting correspondence into the total vote for each alternative: $T : V \rightarrow [0,n]^A$. The function E maps the voting correspondence into the seat share for each alternative: $E : V \rightarrow [0,1]^A$. We will assume that seats are infinitely divisible, to abstract from rounding problems.

We can define the following properties of the seat share function E.

Anonymity: Let σ be a function that permutes N. Then E is anonymous if $E(V) = E(\sigma V)$.

Neutrality: Let π be a function that permutes A. Then E is neutral if $\pi E(V) = E(\pi V)$.

Cancellation property: If $T_j = T_k$, then $E_j = E_k$. (If the vote share for two alternatives is the same, then their seat shares must be the same.)

Nonnegative (positive) responsiveness: Let V' be a vote pattern over $N \times A$. Let $V'' = V'$, except that some voters or abstainers have switched to alternative j:

$$({}_iV'_j \Rightarrow {}_iV''_j; \exists\, i \in N : {}_iV''_j \wedge \sim {}_iV'_j; \forall\, i \in N : \sim {}_iV''_j, \forall\, k \in A, {}_iV'_k \Leftrightarrow {}_iV''_k).$$

Function E is nonnegatively responsive iff

$$\forall \quad (j,k \in A : k \neq j)\ E_j(V') = E_i(V') \Rightarrow E_j(V'') \geq E_i(V''),$$

and E is positive responsive iff $\forall\ (j,k \in A : k \neq j)\ E_j(V') = E_i(V') \Rightarrow E_j(V'') > E_i(V'')$.

Weak plurality ranking property: $T_{j\in A} > T_{k\in A} \Rightarrow E_j \geq E_k$. (If alternative j wins more votes than alternative k, alternative j receives a seat share greater than or equal to alternative k.)

Strong plurality ranking property: $T_{j\in A} > T_{k\in A} \Rightarrow E_j > E_k$. (If alternative j wins more votes than alternative k, alternative j receives a greater seat share than alternative k.)

LEMMA 1: *Any anonymous and neutral seat share allocation function satisfies the cancellation property.*

PROOF: The proof is isomorphic to the proof of Lemma 2 in Hout, Swart, and Veer (2002). We need to show that

$$E_{j\in A} = E_{k\in A} \quad \text{for all } i, k \in P \quad \text{when } T_j = T_k \quad \text{and anonymity and neutrality hold.}$$

If $T_j = T_k$, we can permute all voters so that the voters of alternatives j and k change places. We will call the permutation function σ'. By anonymity, the seat shares stay the same: $E_j(\sigma' V) = E_j(V), E_k(\sigma' V) = E_k(V)$. We can permute the parties so that alternatives j and k change places. We call this permutation function π'. By neutrality, the seat share again remains the same: $E_j(\pi' \sigma' V) = E_k(\sigma' V)$. However, by construction $\pi' \sigma' V = V$. (We have swapped the supporters of alternatives j and k, and then swapped the names of the parties, so we are back to the original situation.) Therefore, $E_j(\pi' \sigma' V) = E_j(V) = E_k(\sigma' V) = E_k(V)$. QED

PROPOSITION 1: *Any seat share allocation rule that is anonymous, neutral, and positively (nonnegatively) responsive satisfies the strong (weak) plurality ranking property.*

PROOF: The proof is isomorphic to the proof of Theorem 2 in Hout, Swart, and Veer (2002). For the positively responsive case, we need to show that neutrality, anonymity, and positive responsiveness imply $T_{j \in A} > T_{k \in A} \Rightarrow E_j > E_k$. For the negatively responsive case, we need to show that neutrality, anonymity, and positive responsiveness imply $T_{j \in A} > T_{k \in A} \Rightarrow E_j \geq E_k$.

By Lemma 1, anonymity and neutrality imply the cancellation property:

$$T_j = T_k \Rightarrow E_j = E_k.$$

Let V″ be a vote pattern where $T_j > T_k$. Then we can derive a vote pattern V' where $T_j = T_k$ by having an appropriate number of voters for alternative j abstain, so that the votes for alternatives j and k are equal.

By the cancellation property, $E_j(V') = E_k(V')$.

By positive responsiveness, $E_j(V') = E_k(V') \Rightarrow E_j(V'') > E_k(V'')$.

By nonnegative responsiveness, $E_j(V') = E_k(V') \Rightarrow E_j(V'') \geq E_k(V'')$. QED

EXTENSION TO COALITIONS

We may arbitarily partition the set of alternatives A into m nonoverlapping coalitions, $c_1 \ldots c_m$, where $1 \leq m \leq a$. Let us define the set of coalitions resulting from such a partition as C, where

$$C = \{c_1, \ldots, c_m\} : c_i \subseteq A; \qquad (\forall\, i, j : i \neq j)\ c_i \cap c_j = \varnothing; \qquad \bigcup_m c_i = A.$$

Let us define the coalition vote correspondence K on $N \times C$, in terms of the vote correspondence for alternatives, V. Let ${}_{i \in N}K_{c \in C}$ read "i votes for an alternative in coalition c." Thus ${}_{i \in N}K_{c_k \in C} \Leftrightarrow {}_{i \in N}V_{j \in c_k}$.

Let us define the coalition vote total function for partition C as the sum of the votes received by each alternative in the coalition. Thus

$$U : K \rightarrow [0,n]^m; \qquad U_k = \sum_{i \in N} {}_iK_k = \sum_{j \in c_k} \sum_{i \in N} {}_iV_j, \quad \text{where } k = 1 \ldots m.$$

Let us define the coalition seat share function for partition C as the sum of the seats allocated to each alternative in the coalition. Thus

$$F : K \rightarrow [0,1]^m; \qquad F_k = \sum_{j \in c_k} E_j(V), \quad \text{where } k = 1 \ldots m.$$

We can redefine the properties used in Proposition 1 for use with coalitions: A seat allocation function E is anonymous (neutral, positively responsive, nonnegatively responsive, canceling, weakly plurality ranking, strongly plurality ranking) for coalitions iff, for all partitions C, the coalition seat allocation function F derived from it is anonymous (neutral, positively responsive, nonnegatively responsive, canceling, weakly plurality ranking, strongly plurality ranking).

Anonymity: A coalition seat allocation function F is anonymous iff $F(K) = F(\sigma K)$, where σ is a function that permutes N.

Neutrality: A coalition seat allocation function F is neutral iff $\tau F(K) = F(\tau K)$, where τ is a function that permutes C.

Cancellation property: A coalition seat share allocation function is canceling iff $U_{j \in P} = U_{k \in P} \Rightarrow F_j = F_k$. (If the vote totals for two coalitions are the same, then their seat shares must be the same.)

Nonnegative (positive) responsiveness in coalition seat share allocation function F can be defined as follows: Let K' be a vote pattern over $N \times C$. Let $K'' = K'$, except that some voters have switched to parties in coalition j:

$$({}_iK'_j \Rightarrow {}_iK''_j; \exists\, i \in N : {}_iK''_j \wedge \sim {}_iK'_j; \forall\, i \in N : \sim {}_iK''_j, \forall\, k \in C, {}_iK'_k \Leftrightarrow {}_iK''_k).$$

Function F is nonnegatively responsive iff $\forall\ (j, k \in C : k \neq j)\ F_j(K') = F_k(K') \Rightarrow F_j(K'') \geq F_k(K'')$, and F is positively responsive iff $\forall\ (j, k \in C : k \neq j)\ F_j(K') = F_k(K') \Rightarrow F_j(K'') > F_k(K'')$.

Weak (strong) plurality ranking property: F satisfies the weak plurality ranking property iff $U_{j \in A} > \mathrm{U}_{k \in A} \Rightarrow F_j \geq F_k$. (If the alternatives in coalition j win more votes than the alternatives in coalition k, coalition j receives an aggregate seat share greater than or equal to coalition k.) Function F satisfies the strong plurality ranking property iff $U_{j \in A} > \mathrm{U}_{k \in A} \Rightarrow F_j > F_k$. (If the alternatives in coalition j win more votes than the alternatives in coalition k, coalition j receives a greater aggregate seat share than coalition k.)

LEMMA 2: *Any anonymous and neutral coalition seat share allocation function satisfies the cancellation property.*

PROOF: Isomorphic to Lemma 1.

LEMMA 3: *Any coalition seat share allocation rule that is anonymous, neutral, and positively (nonnegatively) responsive satisfies the strong (weak) plurality ranking property.*

PROOF: Isomorphic to Proposition 1.

PROPOSITION 2: *Any seat share allocation rule that is anonymous, neutral, and positively (nonnegatively) responsive for coalitions satisfies the strong (weak) plurality ranking property for coalitions.*

PROOF: If seat share allocation function E is anonymous, neutral, and positively (nonnegatively) responsive for coalitions, then, for any partition C, by definition the coalition seat share function F derived from it must be anonymous, neutral, and positively (nonnegatively) responsive.

By Lemma 3, if F is anonymous, neutral, and positively (nonnegatively) responsive, it must satisfy the strong (weak) plurality ranking property. If, for any parition C, F satisfies the strong (weak) plurality ranking property, then by definition E satisfies the strong (weak) plurality ranking property for coalitions. QED

MAJORITY RULE COALITION FORMATION

Let us define pure list proportion representation as a seat share allocation function that allocates a seat share (a real number in [0,1] to each alternative that is equal to the share of the total vote that alternative won).

Coalition games can be defined in terms of the set of winning coalitions $W \subseteq C$. Under majority rule, a coalition is winning iff it has a majority of seat share: $c \in W$ iff $F_c > F_{A-c}$. Under pure list proportional representation, a coalition wins a majority of seat share iff it has more than 50 percent of the vote: $c \in W$ iff $U_c > U_{A-c}$.

PROPOSITION 3: *Any seat share allocation function that is anonymous, neutral, and positively (nonnegatively) responsive for coalitions defines a majority rule coalition game with a set of winning coalitions that is identical to (a subset of) that defined by seat share allocation by pure proportional representation.*

PROOF:

Nonnegative responsiveness: Given that the set of majority rule winning coalitions is defined as $\{c \in C : F_c > F_{A-c}\}$, and under pure list proportional representation the set of coalitions with a majority of seat share is $\{c \in C : U_c > U_{A-c}\}$, we need to show that if seat allocation function E is anonymous, neutral, and nonnegatively responsive for coalitions, then

$$\forall \quad c \in C : U_c > U_{A-c} \Leftarrow F_c > F_{A-c}.$$

If the seat allocation function E is anonymous, neutral, and nonnegatively responsive for coalitions, then by Proposition 2 it satisfies the weak plurality ranking property for coalitions, and thus the coalition seat share allocation F derived from it must satisfy the weak plurality ranking property, $U_{j\in A} > U_{k\in A} \Rightarrow F_j \geq F_k$.

Suppose $F_c > F_{A-c}$ but $U_c \leq U_{A-c}$. If $U_c < U_{A-c}$ then by the weak plurality ranking property, $F_c \leq F_{A-c}$. Contradiction. If E is anonymous and neutral for coalitions, then F must be anonymous and neutral and thus by Lemma 2 must satisfy the cancellation property. If $U_c = U_{A-c}$, then by the cancellation property $F_c = F_{A-c}$. Contradiction. QED

Positive responsiveness: We need to show that if seat allocation function E is anonymous, neutral, and positively responsive for coalitions, then $\forall\ (c \in C)\ U_c > U_{A-c} \Leftrightarrow F_c > F_{A-c}$.

Given that positive responsiveness implies nonnegative responsiveness, we have already shown that $\forall\ (c \in C)\ U_c > U_{A-c} \Leftarrow F_c > F_{A-c}$. All that remains is to show that $\forall\ (c \in C)\ U_c > U_{A-c} \Rightarrow F_c > F_{A-c}$. This is simply the strong plurality ranking property for F. If the seat allocation function E is anonymous, neutral, and positively responsive for coalitions, then by Proposition 2 it satisfies the strong plurality ranking property for coalitions, and thus the coalition seat share allocation F derived from it must satisfy the strong plurality ranking property. QED