TWELVE

Bishop Berkeley Exorcises the Infinite

It all began simply enough when Molyneux asked the wonderful question whether a person born blind, now able to see, would recognize by sight what he or she knew by touch (Davis 1960). After George Berkeley elaborated an answer, that we learn to perceive by means of heuristics, the foundations of contemporary mathematics were in ruins. Contemporary mathematicians waved their hands and changed the subject.[1] Berkeley's answer received a much more positive response from economists. Adam Smith, in particular, seized upon Berkeley's doctrine that we learn to perceive distance to build an elaborate system in which one learns to perceive one's self-interest.[2] Perhaps because older histories of mathematics are a positive hindrance in helping us understand the importance of Berkeley's argument against infinitesimals,[3] its consequences for *economics* have passed unnoticed. If infinitesimal numbers are ruled

1. The mathematically decisive event that changed the situation and let historians appreciate the past was Abraham Robinson's development of nonstandard analysis. "The vigorous attack directed by Berkeley against the foundations of the Calculus in the forms then proposed is, in the first place, a brilliant exposure of their logical inconsistencies. But in criticizing infinitesimals of all kinds, English or continental, Berkeley also quotes with approval a passage in which Locke rejects the actual infinite. . . . It is in fact not surprising that a philosopher in whose system perception plays the central role, should have been unwilling to accept infinitary entities" (Robinson 1974, 280–81). Robinson's appreciation of Berkeley has yet to be fully taken into account by philosophers. For example, Gabriel Moked's discussion of Berkeley's doctrine of strict finitism might have considered Berkeley's claim that mathematics can do without infinitesimals to ask what sort of mathematics results (1988).

2. Smith has been criticized by Salim Rashid—and not without evidence—for insufficient acknowledgment of his intellectual debts (1998). Smith's debt to Berkeley is paid with the customary coin of scholarship (*Essays,* 148): "Dr. Berkeley, in his New Theory of Vision, one of the finest examples of philosophical analysis that is to be found, either in our own, or in any other language, has explained, so very distinctly, the nature of the objects of Sight. . . . Whatever I shall say upon it, if not directly borrowed from him, has at least been suggested by what he has already said."

3. "The last chapter [of *Non-standard Analysis*] contains a review of certain stages in the history of the Differential and Integral Calculus that had to do with the theory of infinitesimals. The fact that the more recent writers in this field were convinced that no such theory can be developed effectively, colored their historical judgment. Thus, a revision has now become necessary" (Robinson 1974, 4).

out, who cares? But we wish to maintain an algebra with division, so we rule out infinite numbers, which were conventionally employed to represent the cost of Hell and the benefit of Heaven. *This* implication of Berkeley's argument has a dramatic implication for our understanding of the basis of the antislavery coalition, which included both utilitarians who believed in Heaven and Hell and those who did not.

The social doctrine set forth by John Locke, outside the *Two Treatises,* depends critically upon individual belief about possible states of infinite pain or pleasure. The critical reason for Locke's refusal to tolerate atheism is his claim that without belief in the infinite bliss of Heaven, foregone by a criminal, there is no reason to think a calculating atheist will avoid crime (Levy 1982b, 1992). In Locke's formulation, there can be no basis of agreement between believers and unbelievers.

In the discussion of religion by David Hume and Smith, questions of the substance of belief—the infinite worth of Heaven—have largely vanished. What happened between Locke and Hume-Smith? One answer is Pierre Bayle and Bernard Mandeville (Levy 1992). Berkeley's doctrine that only the finite is perceived is, I think, a better answer. Out of sight, out of mind.

I propose to give a close look at the content and consequences of Bishop Berkeley's once famous *Towards a New Theory of Vision.* While there are other aspects of Berkeley's work that attract attention from historians of economics (Levy 1982a; Rashid 1990), I claimed in chapter 9 that Berkeley's insight into human perception in *Theory of Vision* is central to Adam Smith's attempt to found economic behavior on the self-awareness of systematic illusion. There is no doubt that Smith finds much of human behavior characterized by illusion. The difficulty economists have with these arguments arises, or so it seems to me, from a modern unwillingness to believe that illusion can be systematic.

There are four pieces to my argument. First, we consider Berkeley's statement of what we shall call his doctrine of strict finitism in perception.[4] Berkeley puts forward two postulates: (1) that there exists some minimum perceptible quantity and (2) that distance is not perceived directly but by experience. Second, we define strict finitism and demonstrate that it translates into the claim that perception is fuzzy in a technical sense. If Berkeleyan perception is fuzzy, then some modern criticism of Berkeley's strict finitism is based upon a simple misunderstanding of the properties of fuzzy relations. This is quite easy to set right. Getting right the doctrine of those who implicitly take perception to be fuzzy (Adam Smith is my prime candidate) will be much harder. Third,

4. I hope this does not clash with established usage. There are two possibilities of confusion of which I am aware. Wittgenstein's doctrine in *Tractatus* sometimes goes by this name. For example, his number theory is criticized by Russell (1971, xx) on this ground. Finitary methods are employed in mathematics when infinity is accepted only as a potential (Kleene 1971, 62–63).

we reconsider Berkeley's dispute with Mandeville on the role of infinite gain in choice. Here, we find Mandeville putting forward the Berkeleyan position that if infinite amounts mattered people would not behave as they do. Berkeley's position against Mandeville, however, seems to depend upon the supposition that infinite distance is sensible.[5] Earlier, however, the position that Mandeville argued for was defended by Berkeley.

Fourth, we ask why all this is not well known. The debate between Berkeley and Mandeville has been well studied, as has been the link between Berkeley and the Scots. In fact, our work focuses on one aspect of Berkeley's work that leads to his challenge of the contemporary foundations of the calculus. If sense is a matter of perception and infinitesimals cannot be perceived, then they are quite literally nonsense. Armed with this insight, Berkeley refuted contemporary mathematics. Nevertheless, the calculus was too valuable to give up. After, but only after, the infinitesimal calculus had been put on secure foundations, it was costless for mathematicians to acknowledge that Berkeley's criticism was correct. By delaying this acknowledgment, they gave up little except the reputation of those long dead. Before the repairs were made, they would have had to give up calculus to maintain consistency.[6] The unfortunate consequence is that the formal aspects of Berkeley's insight about perception have been buried. There is a wonderful irony in the fact that the mathematical work of a founding father of pragmatism can be pointed to as an exemplar of deflection from falsification allowed in the modern pragmatic tradition that mixes together elements of desire and inference.[7]

I suspect that only after the high seriousness of Berkeley's destructive arguments are appreciated will scholars be willing to make the effort to struggle

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5. Berkeley's difficulties with maintaining consistency between the strict finitism of his mathematics and the free and easy use of infinities in Christian doctrine has been noted before (Belfrage 1987, 51–55). As far as I know, this difficulty has not been seen in Berkeley's controversy with Mandeville, as an example of such blindness (Levy 1982b).

6. In wonderful confirmation of the fact that after revolutions textbooks acquire new heroes (Kuhn 1962, 137), Berkeley has made his appearance in a calculus text! "All three approaches had serious inconsistencies which were criticized most effectively by Bishop Berkeley in 1734. However, a precise treatment of the calculus was beyond the state of the art at the time, and the intuitive descriptions . . . of the derivative competed with each other for the next two hundred years" (Keisler 1976, 874).

7. C. S. Peirce (1955, 269): "Any philosophical doctrine that should be completely new could hardly fail to prove completely false; but the rivulets at the head of the river of pragmatism are easily traced back to almost any desired antiquity. Socrates bathed in these waters. Aristotle rejoices when he can find them. They run, where least one would suspect them, beneath the dry rubbish-heaps of Spinoza. Those clean definitions that strew the pages of the *Essay concerning Humane Understanding* (I refuse to reform the spelling) had been washed out in the same pure spring. It was this medium, and not tar-water, that gave health and strength to Berkeley's earlier works, his *Theory of Vision* and what remains of his *Principles*."

with the positive consequences of his doctrine. It is a remarkable coincidence that both nonstandard analysis and fuzzy set theory are products of the 1960s.[8] Nonstandard analysis has knocked down the technical barriers to appreciating the destructive force of Berkeley's doctrine; fuzzy set theory may open the technical door to appreciating his positive doctrine.

The Mediation of Mathematical Concepts by Perception

Pragmatism is the philosophical doctrine holding that desire influences the ways we think of and use language.[9] Berkeley's avowed intention is to place the semiotics of vision upon a pragmatic foundation.[10] To this end, he asks how the mathematical language of distance is mediated by perception.

The first of Berkeley's principles is that distance is not perceived directly; we learn to interpret sense data in terms of size and distance. Thus,

> it is plain that distance is in its own nature imperceptible, and yet it is perceived by sight. It remains, therefore, that it be brought into view by means of some other idea that is itself immediately perceived in the act of vision. (1975, 10)

Berkeley's account is that the fuzziness of an object—fuzzy in a nontechnical sense—is interpreted as a key to its distance:[11]

> [A]n object placed at a certain distance from the eye, to which the breadth of the pupil bears a considerable proportion, being made to approach, is seen more confusedly: and the nearer it is brought the more confused

8. A useful bibliography of work in fuzzy set theory is found in Klir and Folger 1988.

9. "Let us therefore try to get an idea of a human logic which shall not attempt to be reducible to formal logic. Logic, we may agree, is concerned not with what men actually believe, but what they ought to believe, or what it would be reasonable to believe. . . . [T]he highest ideal would be always to have a true opinion and be certain of it; but this ideal is more suited to God than to man. We have therefore to consider the human mind and what is the most we can ask of it. The human mind works essentially according to general rules or habits; . . . We can therefore state the problem of the ideal as 'What habits in a general sense would it be best for the human mind to have? This is a kind of pragmatism: we judge mental habits by whether they work, i.e. whether the opinions they lead to are for the most part true, or more often true than those which alternative habits would lead to. Induction is such a useful habit, and so to adopt it is reasonable. All that philosophy can do is to analyse it, determine the degree of its utility, and find on what characteristics of nature this depends" (Ramsey 1990, 89–90, 93–94).

10. "Upon the whole, I think we may fairly conclude that the proper objects of vision constitute a universal language of the Author of Nature, whereby we are instructed how to regulate our actions in order to attain those things that are necessary to the preservation and well-being of our bodies, as also to avoid whatever may be hurtful and destructive of them" (Berkeley 1975, 51–52).

11. The implication here is that vision in a world without dust, for example, on the space shuttle, would require retraining since the "further is fuzzier" rule would not hold.

> appearance it makes. And this being found constantly to be so, there ariseth in the mind an habitual connexion between the several degrees of confusion and distance; the greater confusion still implying the lesser distance, and the lesser confusion the greater distance of the object. (12)

Vision gives signals that must be interpreted. Berkeley sketches how the semiotics of morals would work:

> Nor doth it avail to say there is not any necessary connexion between confused vision and distance, great or small. For I ask any man what necessary connexion he sees between the redness of a blush and shame? And yet no sooner shall he behold that colour to arise in the face of another, but it brings into his mind the idea of that passion which hath been observed to accompany it. (12)

Berkeley claims that our perception of magnitude and distance are mixed:

> I have now done with distance, and proceed to shew how it is that we perceive by sight the magnitude of objects. It is the opinion of some that we do it by angles, or by angles in conjunction with distance: but neither angles nor distance being perceivable by sight, and the things we see being in truth at no distance from us, it follows that as we have shewn lines and angles not to be the medium the mind makes use of in appending the apparent place, so neither are they the medium whereby it apprehends the apparent magnitude of objects.
>
> It is well known that the same extension at a near distance shall subtend a greater angle, and at a further distance a lesser angle. And by this principle (we are told) the mind estimates the magnitude of an object, comparing the angle under which it is seen with its distance, and thence inferring the magnitude thereof. What inclines men to this mistake . . . is that the same perceptions or ideas which suggest distance do also suggest magnitude. But if we examine it we shall find they suggest the latter as immediate as the former. (22–23)

To identify an object we must separate distance and magnitude. It is in this context that Berkeley claims that there is a minimum visible object:

> It hath been shewn there are two sorts of objects apprehended by sight; each whereof hath its distinct magnitude, or extension. The one, property tangible, i.e. to be perceived and measured by touch, and not immediately falling under the sense of seeing: the other, properly and immediately visible, by mediation of which the former is brought into view. Each of these

> magnitudes are greater or lesser, according as they contain in them more or fewer points, they being made up of points of minimums. For, whatever may be said of extension in abstract, it is certain sensible extension is not infinitely divisible. There is a *Minimum Tangible* and a *Minimum Visible,* beyond which sense cannot perceive. This everyone's experience will inform him. (Berkeley 1975, 23)

Berkeley often refers to this minimum as a "mite," abbreviated in his *Philosophical Commentaries* as "M."[12]

The hypothesis that our perception is bounded at some finite M is completely innocuous if it is independent of distance. If we perceive the same M at a distance of one meter, as we do between here and the moons of Jupiter, we have gutted the assumption of any substance. Berkeley knows this and blocks it with the following claim:

> Now for any object to contain several distinct visible parts, and at the same time be a *minimum visible,* is a manifest contradiction.
>
> Of these visible points we see at all times an equal number. It is every whit as great when our view is contracted and bounded by near objects as when it is extended to larger and remoter. For it being impossible that one *minimum visible* should observe or keep out of sight more than one other, it is a plain consequence that when my view is on all sides bounded by the walls of my study, I see just as many visible points as I could, in case that the removal of the study-walls and all other obstructions, I had a full prospect of the circumjacent fields, mountains, sea, and open firmament: for so long as I am shut up within the walls, by their interposition every point of the external objects is covered from my view: but each point that is seen being able to cover to exclude from sight one only other corresponding point, it follows that whilst my sight is confined to those narrow walls I see as many points, or *minima visibilia,* as I should were those walls away, by looking on all the external objects whose prospect is intercepted by them. (Berkeley 1975, 33)

This argument can be expressed geometrically. Suppose that at some unit of distance along the horizontal, Δ, the minimum perceptible magnitude along the vertical is M. Berkeley argues that if at some status quo M can block some X from view, X must be a minimum visible too. Assuming linearity at 1,000Δ

12. Berkeley's mathematical conjecture, soon to be considered in more detail, is that M could serve to refound calculus (1975, 281 [B333]): "Newton's fluxions needless, any thing below an M. might serve for Leibnitz's Differential Calculus." This passage is not considered by Moked (1988) in his list of *Philosophical Commentary* passages that deal with minima.

the minimum visible is 1,000M. We can solve for the angle Θ as the minimum angle of perception; Θ is simply arctan M/Δ.

If visual sense is based on angles, then the problem for the perceiving subject is to distinguish between distance and size. In the term that econometricians coined, this is the *classical identification problem.* We have one sense datum, Θ, and two unknowns, size and distance. Looking at the problem this way, it is clear why touch is important for Berkeley's argument. Touch gives another piece of information to help with the object's identification. With touch, there are two pieces of sense data and two unknowns. A problem with two observations and two unknowns is a good deal more promising than a problem with one observation and two unknowns.

Outside the range of touch, then what? We learn to infer that the moon is farther away than a cloud because a cloud can cover the moon. Ideas allow us to interpret perception. This is why, in my interpretation, Smith's theory of conduct is intertwined with ideas. Rules, heuristics, guide inferences. This interpretation of Berkeley's argument demonstrates why Samuel Bailey's objections did not sway Berkeley's admirers. J. S. Mill and James Ferrier's defense of Berkeley's doctrine point to the identification problem.[13]

Fuzzy Perception

Berkeley's account concerns vision over physical distance. As we have seen, Smith generalizes this principle in his spectator theory of morality. If we cannot judge distance over material income perfectly, then what reason do we have to believe that we can pursue our material self-interest correctly? A doctrine of imperfect perception makes it easier to appreciate why individuals might prefer constraining rules of conduct to choice guided purely by perception of self-interest. The rules we have encountered above are the Golden Rule of Christianity and the Greatest Happiness Principle of Utilitarianism. When percep-

13. "[I]f a child fancies the moon to be no larger than a cheese, it is because he forgets that it is farther off, and draws from the visual appearance an inference which would be well grounded if the moon and the cheese were really at an equal distance from him" (Mill 1842, 321). Ferrier (1988, 839) works through the identification problem, pointing out the problem of distinguishing size and distance. One part of Bailey's objection to Berkeley is strange because it seems to rest on a property of Berkeley's model that Berkeley nowhere defends: "Berkeley's theory takes for granted, that when we see objects at various distances, those distances, or in other words, the intervening tangible spaces between us and the objects, are suggested to the mind" (1988, 84–85).

Mill is also puzzled, observing that "we see bodies and their distances by precisely the same mechanism. We see two stars, if they are imaged on the retina, and not otherwise; we see the interval between those stars if there is an interval on the retina between two images, and if there is no such interval we see it not. . . . Surely this argument does not depend upon an implied assumption that the intervals between objects are physical lines joining them." (1843, 492).

tion fails, rules may serve as substitutes (Levy 1992). To this end, we consider, somewhat seriously, what an individual believes about distance.

Following Smith's lead, we consider not only distance over space but distance over commodities. In the standard (sharp) account of choice, in a one-commodity world, preferences over commodity space flow automatically from commodity space itself since for any bundles $a, b, c \in R^1$ if $a > b$ then $P(a,b)$ holds and conversely.[14] In this context, indifference in choice between *a* and *b*, $I(a,b)$, follows from equality of *a* and *b*. The standard (sharp) axioms of transitivity of *P* and *I* and completeness over R^N with respect to *P* and *I* are unnecessary in R^1 since they follow from the familiar facts about the relations "greater than" and "equal." Our modification is to suggest that preferences over *b* and *a*, $P(b,a)$, follow from the perception over commodity space rather than from commodity space itself. To this end, we introduce the relation $BB(b,a|sq)$ to signify the belief, at the status quo, that is at *sq*, *b* is greater than *a*.

We write down the details of *BB*, where as far as possible the argument shall proceed by a reinterpretation of standard assumptions. For instance, we know in the standard account for all $b \in R^1$ there exists a *b*-roof and *b*-floor, which we write as $\overline{b}$ and $\underline{b}$ which are respectively the greatest lower bound of all elements of R^1 perceived to be strictly larger than *b* and the least upper bound of all elements of R^1 perceived to be strictly less than *b*. These names are imported from computer science where considerations of finite precision of computations are paramount (Iverson 1962, 12). In standard (sharp) consumer theory, we can easily prove that both $\overline{b}$ and $\underline{b}$ exist and are equal to *b*. Where our account shall diverge from the standard one is the specification that the difference between $\overline{b}$ and $\underline{b}$ is noninfinitesimal.

It will be quite convenient to take the truth value of *BB* to be a function of *a*, *b*, and *sq*. The way this can be expressed in the standard (sharp) account is $BB(a,b|sq)$ holds (is true or equals 1) if and only if $a > b$, regardless of the state of *sq*.

$$BB(a,b|sq) = 0 \text{ if } a \leq b$$

$$BB(a,b|sq) = 1 \text{ if } a > b, \forall\, sq \in R^1$$

Using 1 and 0 as marks of truth and falsity is convenient. Binary theory conditions are indicated by allowing the set of truth values to describe *BB* to be {0,1}.

What is required of $BB(a,b|sq)$ is the following:

Axiom 1. $BB(a,b|sq) = \Phi \in [0,1]\ \forall\, a, b, sq \in R^1$

14. The relational notation employed follows that of Robinson 1974.

This formulation allows for the possibility that truth values are fuzzy, that is, they take on values outside {0,1} (Kaufmann 1975). The case of sharp truth values is consistent with axiom 1 because {0,1} is a special case of [0,1].

These fuzzy truth values can be given an objective probabilistic interpretation. We could interpret the claim that $BB(a,b|sq)$ has a truth value Φ as the claim that standing from *sq* we would believe *a* to be bigger than *b* with relative frequency Φ. Thus, fuzziness is akin to indifference in that if given a choice between *a* and *b* sometimes one picks *a* and sometimes one picks *b*. Allowing indifference to range between 0 and 1 obviates any requirement that observed relative frequency of choice between *a* and *b* be only zero, 100, or 50 percent (fig. 6).

Figure 6 gives a picture of the perception from the status quo *sq* of *b*. Any amount at or above $\bar{b}$ is perceived clearly to be bigger than *b*; any amount at or below $\underline{b}$ is perceived to be smaller than *b*. Inside the open interval $(\underline{b}, \bar{b})$ perception is fuzzy; thus, Φ can be anywhere on the [0,1] interval. If perceptions were sharp, the judgment *BB* would jump from a Φ of 0 to one of 1 at *b*, as indicated by the dotted line.

We assume the existence of these roof and floor bounds:

Axiom 2. $BB(a,b|sq) = 0$ if $a \leq \underline{b}$

$BB(a,b|sq) = 1$ if $a > \bar{b}$ $\exists\, \underline{b}, \bar{b} \in R^1$ $\forall\, a, b, sq \in R^1$.

If $\bar{b}$ and $\underline{b}$ are both *b*, then axiom 2 collapses to the sharp version. In terms of the relation *BB*, we can then define the principle of strict finitism.

Strict finitism requires the following: (1) there exists some positive finite M such that if something less than M is added (or subtracted) from any point $a \in R^1$ relative to a fixed *b* and a status quo *sq* for any individual, *no one* can perceive the difference; and (2) a *BB* relation with a truth value of one can be perceived by an outside observer to differ from a relation with a truth value of zero.

Theorem. Strict finitism implies that the relation *BB* takes on fuzzy truth values.

Proof. Without loss of generality, we take Φ to be a function of a variable *a* with respect to fixed *b* and *sq*. Assume the contrary of what is to be proved, that truth values of *BB* are restricted to {0,1}. Then, we can find some *a* where the addition of some amount smaller than M would change $\Phi(a)$ from zero to one. The existence of this *a* is guaranteed by the suppositions (1) that truth values are dichotomous and (2) that at some point more will be perceived to differ from some fixed lesser point. The outside observer can detect a change in $\Phi(a)$ from zero to one. Thus, an imperceptible change in *a* has a perceptible consequence: the change in Φ from zero to one. This contradicts strict finitism. Thus, the change in $\Phi(a)$ must itself be imperceptible and be smaller than the

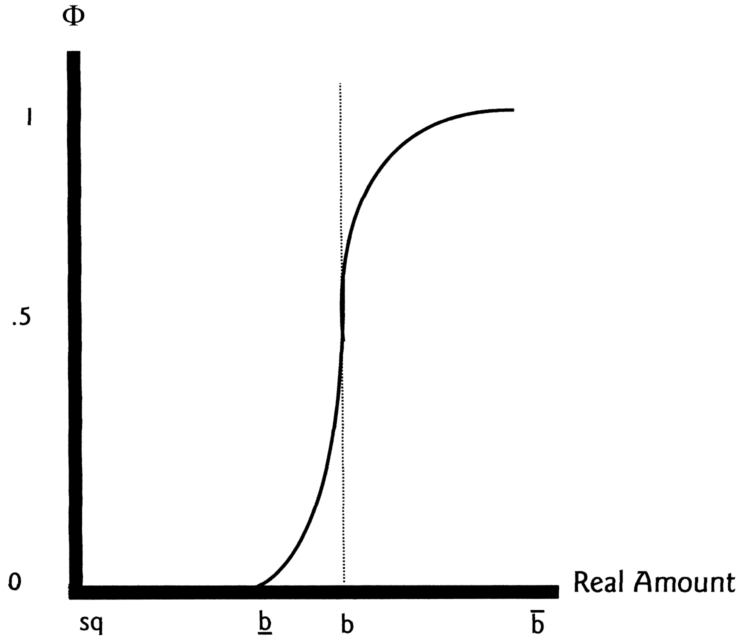

Fig. 6. Believed bigger than *b* at *sq*

change from zero to one. This establishes that Φ(*a*) takes on fuzzy truth values.

Given that *BB* has fuzzy truth values, or is a fuzzy relation, we can immediately free Berkeley's construction from recent charges of inconsistency. One defining feature of fuzzy relations is that they are not transitive. That is to say, consider some heap of coffee composed of a very large number of very small grains. Consider the no discernible difference relation in perception and let us suppose that no one grain makes a difference. Step by step, one removes grains. Obviously, after most of the heap is removed one can distinguish the starting point from the case of the almost removed heap *even though* at no point in the procedure could one distinguish a heap with one grain more or less. Thus, the no discernible difference in perception relation is not transitive. It is indeed part of the attraction of fuzzy mathematics to deal with the discomfort first expressed by Henri Poincaré that only in mathematics is equality transitive.

If Berkeley is describing fuzzy perceptions, then it is quite beside the point

15. Armstrong (1960, 43) describes "difficulties" with the doctrine of the minimum perceptible: "We may put a dilemma: either these minima have an extension ('the smallest'), or they do not. If they *do* have an extension, it is not difficult to show that they will have parts, that is, they will

to assume a perception relation that is transitive. The mathematical relations that are required to describe Berkeley's minimum of perception will not be transitive.[15]

To make a link between perception and choice, one simply asserts

Axiom 3. $P(a,b|sq) = \mathrm{BB}(a,b|sq)\ \forall\ a, b, sq \in R^1$

This is the fuzzy equivalent of more is preferred to less. It only requires that what is perceived to be more is preferred to what is perceived to be less.

To give the model some structure, we can require that the amount of the perception band, $\bar{b} - \underline{b}$, is a function of the distance between b and the status quo, sq, $d(b,sq)$. Thus, as $d(b,sq)$ increases, $\bar{b} - \underline{b}$ increases. Rule-directed perception will be important for a utilitarian if learning can change the perception band $\bar{b} - \underline{b}$ for a constant sq. The utilitarians we have encountered held the equivalence of the Great Happiness Principle and the Golden Rule.

The utilitarians we have encountered had different views on Heaven and Hell. Suppose that these states have infinite consequences. We consider now how Berkeley's strict finitism purged social analysis of infinites and thus how post-Berkeley Christians would not expect belief in Heaven and Hell to be a decisive explanatory variable in one's behavior.

The vast amount of research on the economics and probability of gambling behavior has been motivated by the observed fact that people do not seem to act to maximize the expected value of a gamble. It is common knowledge that vast numbers of lottery tickets are sold for considerably more than their expected value. Paying a dollar for a one in a million chance to win $200,000 obviously cannot be an instance in which one maximizes the expected value of the action. The expected value of not playing the game ($1.00) is higher than the expected value of playing ($0.20).

To see at a glance how fuzzy perceptions make the assertion that we cannot use expected value considerations to explain gambling less obvious, let us define a *perceived number* $\tilde{b}$ relative to sq as some number selected from the interval $\bar{b} - \underline{b}$. Different people will surely have different perceived numbers. In the case in point, how do we know that for someone with a status quo considerably below $200,000 the vast sum of $1 million is not in fact contained within the perception band of $200,000? If it is, then perhaps the individual gambler is maximizing the *perceived* expected value.[16]

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not be minima. This is easily seen if we consider three adjoining minima, say three in a straight line. By hypothesis, the boundary of the first and the second is separated from the boundary of the second and the third, that is the second minimum has distinguishable boundaries, which implies that after all it *does* have parts." This argument is accepted by Stack (1970, 40–41).

16. An account of gambling as exchange—the *actor* has sharp perception of income but the *spectators* have fuzzy perception over income (and thus approbation from status)—is given in Levy 1999a.

The gambling problem we encounter—gambles with infinite stakes—is known by humanists as Pascal's wager but by economists as the St. Petersburg paradox, although Savage (1972) links gambles with infinity to Pascal. The traditional approach to resolving the St. Petersburg paradox is to bound the utility function, that is, some finite level of income; our axioms deny that vastly more is preferred to less (Samuelson 1977). Our method of dealing with infinity is rather different in spirit. Instead of denying satiation, we require that distant levels of income are heavily depreciated from every status quo. As the status quo changes, the consumer presumably learns to adjust. Instead of assuming satiation, this fuzzy perception formulation requires only a weak type of algebraic closure of the perceived numbers in our theory in conjunction with the principle that there are no perceived infinitesimals.

Start with any two perceived numbers, $\tilde{b}$ and $\tilde{c}$. Suppose they are perceived to be distinct, for example, $(\underline{b},\bar{b}) \cap (\underline{c},\bar{c}) = \varnothing$. Let $\tilde{c} > \tilde{b}$ with $\tilde{b}$ finite. If we wish to allow an algebra in which we can prove that $1 > \tilde{b}/\tilde{c}$, that is, one in which we allow division over the *perceived* numbers, then we know that $\tilde{c}$ cannot be infinite. Assume that it is; thus, the resulting $\tilde{b}/\tilde{c}$ is infinitesimal. But this contradicts the supposition that we cannot perceive infinitesimals. Thus, $\tilde{c}$ cannot be an infinite number.

Algebraic closure might be thought of as an odd requirement. Be that as it may, when we assume that perception is perfect over the real line of commodity space, we automatically assume that perception is closed over algebraic operations over the real numbers.

Of course, Berkeley knew this. Here are some passages from *Philosophical Commentary:*

> An idea cannot exist unperceiv'd. (1975, 285 [B377])
>
> Axiom. No reasoning about things whereof we have no idea. Therefore no reasoning about Infinitesimals. (283 [B354])
>
> We cannot imagine a line or space infinitely great therefore absurd to talk or make propositions about it. (289 [A417])

Mandeville as a Follower of Berkeley

Mandeville's teaching can be seen as a response to some version of Pascal's wager. If people believed what they said about Heaven and Hell, Mandeville asserts, they would not behave the way they do. What they say is wrapped up in notions of infinite gain and infinite loss (Levy 1992).

Here is Pascal's memorable statement of the principle at issue:

> A game is being played at the extremity of this infinite distance where heads or tails will turn up. What will you wager? According to reason, you can do neither the one thing nor the other; according to reason, you can defend neither of the propositions.
>
> Do not then reprove for error those who have made a choice; for you know nothing about it. "No, but I blame them for having made, not this choice, but a choice; for again both he who chooses heads and he who chooses tails are equally at fault, they are both in the wrong. The true course is not to wager at all."
>
> Yes; but you must wager. It is not optional. . . . Your reason is no more shocked in choosing one rather than the other, since you must of necessity choose. This is one point settled. But your happiness? Let us weigh the gain and the loss in wagering that God is. . . . And thus, when one is forced to play, he must renounce reason to preserve his life, rather than risk it for infinite gain, as likely to happen as the loss of nothingness. (1958, 66–67)

To consider Pascal's wager, we require a 2 × 2 matrix that gives us our choice. If we act in accord with God's will, we have a singularly dull existence now, say, receiving zero in material income, but we receive an infinite H later. If we act in accord with what is reported as God's will, but He does not exist, we get zero. If we act contrary to God's will, and He does not really exist, we have a good time, getting positive, albeit finite, material income x. But if God does exist then we must balance the current x with infinite negative $-H$. Matrix 4 gives our choice. From which column do we wish to choose?

MATRIX 4. Pascal's wager

	Act as if God Exists	Act as if God does not exist
In Fact, God Exists	H	$-H + x$
In Fact, God Does Not Exist	0	x

Let us call the probability that God actually exists p. As long as we block infinitesimal probabilities, the strategy of acting as if God exists generates a higher expected value than acting as if God does not exist. That is, for any noninfinitesimal probability p, $0 < p < 1$ the expected value of the game with the universe is

$$p\,H + (1 - p)\,0 > [p(-H + x) + (1 - p)x] \tag{1}$$

This is so because the expression to the left of the inequality is a positive infinite number and one to the right of the inequality is a negative infinite. Needless to say, the argument can be weakened by removing either H or $-H$. We would then compare an infinite with a real number that also can be signed directly. The argument goes through as Pascal claims. The size of x is quite irrelevant; the proof goes through for any finite x. Here is Pascal's statement:

> *The end of this discourse.*—Now, what harm will befall you in taking this side? You will be faithful, honest, humble, grateful, generous, a sincere friend, truthful. Certainly you will not have those poisonous pleasures, glory and luxury; but will you not have others? I will tell you that you will thereby gain in this life, and that, at each step you take on this road, you will see so great certainty of gain, so much nothingness in what you risk, that you will at last recognise that you have wagered for something certain and infinite, for which you have given nothing. (1958, 68)

Now, suppose that instead of dealing with Heaven and Hell directly we work with perceptions of Heaven and Hell. We replace H with $\tilde{H}$ and $-H$ with $-\tilde{H}$ and since these perceptions are finite, the game becomes finite and Pascal's argument fails as an a priori principle. An easy way to see how the division operates is to consider equation 1 in the case without Hell. Thus, let $-\tilde{H} = 0$. Thus, the right-hand side (eq. 1) is x and the left-hand side is $p\tilde{H}$. Divide both sides by $p\tilde{H}$. If $p\tilde{H}$ is infinite, then we generate the forbidden infinitesimal.

What the issue will hinge upon is the details of $\tilde{H}$ and $-\tilde{H}$, the details of the utility function, the actual probabilities, and so on.

Computing Correctly?

Berkeley's response to Mandeville is based on the notion that free thinkers do not compute correctly, just as the response to his free thinking in mathematics is precisely that free thinkers cannot compute correctly.

> *EUPH.* But *Socrates,* who was no Country Parson, suspected your Men of pleasure were such through ignorance.
>
> *LYS.* Ignorance of what?
>
> *EUPH.* Of the art of computing. It was his opinion that Rakes cannot reckon. And that for want of this skill they make wrong judgments about pleasure, on the right choice of which their happiness depends. . . . To make a right computation, shou'd you not consider all the faculties and all the kinds of Pleasure, taking into your account the future as well as the present, and rating them all according to their true value?
>
> *CRI.* The *Epicureans* themselves allowed, that Pleasure which procures a greater Pain, or hinders a greater Pleasure, shou'd be regarded as a Pain; and, that Pain which procures a greater Pleasure, or prevents a greater Pain, is to be accounted a Pleasure. In order therefore to make a true estimate of Pleasure, the great spring of action, and that from which whence the conduct of Life takes its bias, we ought to compute

> intellectual Pleasures and future Pleasures, as well as present and sensible: We ought to make allow in the valuation of each particular Pleasure, for all the Pains and Evils, for all the Disgust, Remorse, and Shame that attend to it . . .
>
> *EUPH.* And all these points duly considered, will not *Socrates* seem to have had reason of his side, which he thought ignorance made Rakes, and particularly their being of what he calls the Science of more and less, greater and smaller, equality and comparison, that is to say of the art of Computing?
>
> *LYS.* All this discourse seems notional. For real abilities of every kind it is well known we have the brightest Men of the age among us. But all those who know the World do calculate that what you call a good Christian, who hath neither a larger Conscience, nor unprejudiced Mind, must be unfit for the affairs of it. Thus you see, while you compute your selves out of pleasure, other compute you out of business. What then are you good for with all computations?
>
> *EUPH.* I have all imaginable respect for the abilities of Free-thinkers. My only fear was, their parts might be too lively for such slow talents as Forecast and Computation, the gifts of ordinary Men. (1732, 1: 119–21)

Mandeville responded to Berkeley with a challenge about infinite gain and infinite loss. If behavior tracked with what we said about infinite gains from Heaven, here is what we would see:

> Since this worldly Greatness is not to be attain'd to without the Vices of Man, I will have Nothing to do with it; since it is impossible to serve God and Mammon, my Choice shall be soon made: No temporal Pleasure can be worth running the Risque of being eternally miserable; and, let who will labour to aggrandise the Nation, I will aim at higher Ends, and take Care of my own Soul.
>
> The Moment such a Thought enters into a Man's Head, all the Poison is taken away from the Book, and every Bee has lost his Sting.
>
> Those who should in Reality prefer Spirituals to Temporals, and be seen to make more Pains attain an everlasting Felicity, than they did for the Enjoyment of the fading Pleasures and transient Glorie of this Life, would not grudge to make some Abatements in the Ease, the Conveniences, and the Comfort of it, or even to part with some of their Possession upon Earth, to make sure of their Inheritance of the Kingdom of Heaven. Whatever Liking they might have to the curious Embellishments and elegant Inventions of the Voluptuous, they would refuse to purchase

> them at the Hazard of Damnation. . . . No Book would be plainer or more intelligible to them than the Gospel; and without consulting either Fathers or Councils, they would be satisfied, that mortifying the Flesh never could signify to indulge every Appetite, not prohibited by an Earthly Legislator. (1953, 22–23)

Mandeville is appealing to a doctrine of strict finitism. Is this possible? Reading Berkeley and Mandeville on the same side of an issue is unusual. Is there evidence? Let us read what Berkeley wrote in his first sermon, "On Immortality":

> Let us but look a little into matter of fact. how far I beseech you do we Xtians surpass ye old Heathen Romans in temperance & fortitude, in honour & integrity? are we less given to pride & avarice, strife & faction than our Pagan Ancestors? With us yt have immortality in view is not ye old doctrine of eat & drink for to morrow we die as much in vogue as ever? We inhabitants of Xtendom enlighten'd with ye light of ye Gospel, instructed by ye Son of God, are we such shining examples of peace and vertue to ye unconverted Gentile world? . . .
>
> I come now to enquire into ye cause of this strange blindness & infatuation of Xtians. whence it is that immortality a happy immortality has so small influence . . . Wherein consists the wondrous mechanism of our passions wch are set a going by the small inconsiderable objects of sense whilst things of infinite weight & moment are altogether ineffectual. (1948–57, 7:10–11)[17]

One explanation is, of course, that the pleasures of Heaven are a great distance away (12–13).

The Pragmatic Response to Berkeley's Mathematical Free Thinking

We have seen how Berkeley offers a pragmatic account of the role of perception across distance. Chief among his premises is his insistence on the irrelevance of the unperceived. Hence, infinitesimals and their algebraic kin are strictly nonsense. Armed with such insight, Berkeley pointed out that at the heart of the calculus wishful thinking was being passed as theorem.

The lynchpin of the pragmatic philosophy of science is the Duhem-Quine thesis, which represents philosophers as rational choosers. All we need to

17. Berkeley (1948–57, 7:12–13) has a version of Pascal's wager.

assume is that theories, such as A and B, have some utility, thus $U(A)$ and $U(B)$, which perhaps depend upon the problems they can solve. A might be the machinery we use to prove that income-compensated demand curves fall, and B might be the specification that the demand curve is log-linear. Suppose we apply A and B to data and discover that the sign is wrong. Now what? One thing we surely will not do is to reject the rationality assumption that gives our sign prediction. So we think a bit more about income effects; voilà, how silly, if we have an inferior good, it cannot be globally log-linear.[18]

The question is whether theory acceptance can be put on a machinelike basis. If so, we need not appeal to pragmatic considerations. To avoid defining away the problem, let us take the operator $\therefore$ to denote "rational acceptance." Popperian philosophy of science emphasizes the logical basic of theory development by focusing on a case in which $\therefore$ would satisfy the deducible relation (Kleene 1971, 90–102). Let $\rightarrow$ denote implication, & conjunction and ~ negation. Commas separate statements. Then for some statements A and X the template of the Popperian $\therefore$ is $A \rightarrow X, \sim X \therefore \sim A$. The attractive feature of the Popperian view of rational acceptance is precisely that there are no subjective elements.

The difficulty is that many interesting theoretical systems are conjunctions. What decision do we make when $(A \,\&\, B) \rightarrow X, \sim X$? The Popperian $\therefore$ only gives us $\sim(A \,\&\, B)$, but suppose we must pick one A or B to drop. What do we do then?

We can look at the Duhem-Quine thesis as extending $\therefore$ to close the ambiguity of the modus tolens in the case of joint hypotheses. We can introduce desire into the center of inference by allowing the following deduction rule for $\therefore$

$$(A \,\&\, B) \rightarrow X, \sim X, U(A) > U(B) \therefore \sim B.$$

This econometric example is harmless because our bag of functional forms is endless. What Berkeley did, however, was not harmless. He showed that the calculus, as presented by Newton and Leibnitz, was inconsistent at its very foundations.[19]

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18. Why not? Because a good cannot be globally inferior if (1) the budget constraint allows zero consumption at zero income and (2) negative consumption is ruled out. As income rises from zero, the good must be normal before it becomes inferior.

19. Robinson (1974, 265–66): "Additional interest is lent to the axiom by the fact that is implies, clearly and immediately, a glaring contradiction, i.e., that the (infinitesimal) differences between two quantities may, at the same time, be both equal to, and different from, zero. . . . Berkeley pointed out some forty years later that the same weakness was present in the method of fluxions. This is true although the absence of a clear basis for that method made it harder to argue against. . . . However, there can be little doubt that the inconsistency in question contributed to the eventual eclipse of the method of infinitely small and infinitely large numbers at the beginning of the nineteenth century."

Infinitesimals were sometimes something and sometimes nothing. If infinitesimal is zero, then

> a + 500 nothings = a + 50 nothings, an innocent, silly truth (Berkeley 1975, 281 [B338]).

But for the calculus to go through, there needs to be some $\Delta > 0$ such that $a + 500\Delta = a + 50\Delta$. How could this be?

A naive response would be "We'll get back to you in two hundred years." This would be very costly because what would replace the calculus in the meantime? The traditional Duhem-Quine position is that an important theory can be saved from empirical contradiction by the sacrifice of an auxiliary aspect of the theory. But what could have saved the calculus if a contradiction were admitted? What stood between the real contradiction and an admitted contradiction? Character assassination, flat denial, and hand waving all seem to have been employed. Character assassination is easy to spot. The hand waving takes the form of the claim: "That is easy to fix, you just do . . ." without bothering to do it. This is noticeable when one can prove that the patch does not work.

Here is a famous response to Berkeley from James Jurin, which proposes a remarkable metamathematical doctrine:

> I do assure you, Sir, from my own experience, and that of many others whom I could name, that the doctrine may be clearly conceived and distinctly comprehended. If *your imagination is strained and puzzled with it,* if it appears to you to contain *obscure and inconceivable mysteries,* in short, if you do not understand it, I tell you others do; and you may do so too, if you will read it with due attention, and with a desire of comprehending it, rather than an inclination to censure it. (1989, 31)

Out of kindness to his intellectual inferiors, Jurin sketched the requisite proof:

> I shall here beg leave, for the sake of readers less mathematically qualified, to put a very easy and familiar case. Suppose two Arithmeticians to be disputing whether vulgar fractions are to be preferred to decimal; would it be fair in him who is for expressing the third part of a farthing by the vulgar 1/3, to affirm that his antagonist proceeding blindfold, and without knowing what he did, when he pretended to express it by 0.33333 &c. because this expression did not give the rigorous, exact value of one third of a farthing? Might not the other reply that, if this expression was not rigorously exact, yet it could not be said he *proceeded blindfold, or without clearness and without science* in using it, because adding more figures he could approach as near as he pleased, he could clearly and distinctly find and demonstrate how much he felt short of the rigorous and exact value? (36)

Later writers amused themselves by debating whether Berkeley was honest but simply stupid in bringing these objections or whether he knew all the time that his objections would vanish like infinitesimals (Cajori 1919, 90).

Let us take a closer look at Jurin's argument and the later reaction to it.

> [T]he second supposed fallacy . . . gives $2x(2y) \div (2y + dy)$. Both of these expressions are equal to $2x$, "which is the result either of two errors or of none at all." If you claim that $2x(2y + dy) \div (2y) > 2x$, how much greater is it, supposing $2x = 1000$ miles? Not as much as the thousand-millionth part of an inch. (Cajori 1919, 68–69)

Although Cajori (91) is troubled by the fact that Jurin offers the argument for popular consumption, he does not seem to appreciate the difficulties which arise in treating *dy* as if it were a small positive real number. The equality holds only the case in which $dy = 0$, which is Berkeley's point. Berkeley depends upon the trichotomy property of real numbers, that is, for any x, one and only one of the following hold: $x > 0$, $x < 0$, $x = 0$. Hence, if $dy = 0$ it cannot be positive too.

The hope behind Jurin's "escape" from Berkeley is to use finite approximations to real number numbers. Of course, this substitution will be needed for computation purposes, but will it pass muster as an account of the foundations of real numbers?

One of the elementary facts about real numbers is that multiplication is associative. Consider two nonzero real numbers, a and x. Then the following holds:

$$(aa^{-1})x = a(a^{-1}x) \tag{2}$$

Suppose, however, that these are just finite approximations to real numbers. Does equation 2 continue to hold? No. The precision with which aa^{-1} can be computed for any nonzero a may vastly exceed the precision with which $a^{-1}x$ can be computed for some x. One way to see the nonassociative property is that we can prove within an infinite precision calculus that $aa^{-1} = 1$. This insight can be used to make finite precision mathematics more closely resemble infinite precision mathematics. The left hand-side of equation 2 can be solved with infinite precision; the right-hand side characteristically cannot.[20] Without the infinite precision calculus as normative guidance, the failure of associative multiplication would mean that any number of answers hold. This

20. The introduction of infinite precision insight in a finite precision environment makes it possible to produce counterexamples to such intuitively obvious propositions as that constraints upon optimization cannot be likelihood enhancing (Levy 1988c).

fact makes numerical computations using a finite-precision computer very interesting (Knuth 1981, 214–30).

After it became clear that Berkeley was not stupid—and calculus in terms of limits had been well-founded—it became interesting to ask why Berkeley said what he did. Let us return to Cajori in a 1985 reprinted edition:

> The publication of Berkeley's *Analyst* was the most spectacular mathematical event of the eighteenth century in England. Practically all British discussions of fluxional concepts of that time involve issues raised by Berkeley. Berkeley's object in writing the *Analyst* was to show that the principles of fluxions are no clearer than those of Christianity. . . . A friend of Berkeley, when on a bed of sickness, refused spiritual consolation, because the great mathematician Halley had convinced him of the inconceivability of the doctrines of Christianity. This induced Berkeley to write the *Analyst.** (1985, 239)

We will consider the footnote indicated by the asterisk—it was supplied by the editor—momentarily.

The most illuminating attempt to explain Berkeley's lack of vision was made by John Wisdom, using methods that leave one breathless. First, Wisdom explains to the reader how the notion of a limit depends upon continuity:

> Despite his superb criticism of Newton's methods of fluxions, Berkeley showed a certain lack of insight into what Newton was trying to do, and this was certainly due to his not being able to grasp the notion, the then current crude notion, of a limit approached by a mathematical function. This notion depends psychologically (though not logically) upon continuity—one thinks of a limit as a goal *towards* which a function *moves* and moves *continuously* . . . Berkeley could not agree that a function could become smaller and smaller indefinitely; any attempt to make it less than a certain minimal size would instantly reduce it to zero; here therefore there was a breach of continuity. In his philosophy also Berkeley denied continuity. (1953, 158–59)

Having established his credentials to criticize Berkeley's mathematics, Wisdom reaches into the Freudian bowl of wisdom:

> *Interpretation XIX: Berkeley's antagonism to the method of fluxions and his attempt to replace it by a method involving discrete quantities were due to his fear that his insides would dissolve into a flux and to his need to have his insides solid, even though this in its turn would prove disturbing.*
>
> This would, of course, imply that there was for Berkeley a faecal element in mathematics itself. (160)

Wisdom gives us his thoughts on infinite divisibility:

> Thus his conception of mathematics banished the notion of flux from very small quantities just as the principles of *Esse percipi* banished Matter. While, however, his conception of mathematics was directly related to repudiation of flux, *Esse percipi* was only indirectly related to it: *Esse percipi* banished the solid poison and, interestingly enough, conveyed the solipsistic picture of a dream-like flux—an example of what Freud has called the "return of the repressed." (161)

We may conclude this chapter by making part of Interpretation XVIII more specific:

> *Interpretation XX: What Berkeley envied in mathematicians was their freedom to regard some faeces as good and to manipulate derivations from these.* (Wisdom 1953, 161)

Is it necessary to remark that Wisdom's book has not been republished? However, Cajori's *History of Mathematics* has been. The asterisk noted earlier was added by the editor of the 1985 edition. The appended note has this to say:

> Berkeley's objections were well taken and could not be dealt with until 1966, with the creation of Non-standard Analysis. Non-standard Analysis has built a new structure of the Calculus within the framework of which infinitesimals are accommodated and in which Berkeley's objections no longer apply. (1985, 490)[21]

So ends the claim of recourse to Christian apologetics, the claim that constipation precludes the understanding of continuity, and Jurin's metamathematical proposal that if three people think the proof is correct it is.

Conclusion

Historians of economics are often reluctant to admit the possibility that our subjects know more than we do. If they knew something that was worth knowing, why has not it been absorbed into modern economics? Perhaps the difficulty is that they did not know how to persuade later economists that they knew something. This is not quite the same as not knowing what they claim to know.

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21. Modern histories of mathematics (Burton 1985, 498–99) are most careful to point out Berkeley's role.