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Introduction

Thomas More's Utopians enjoyed their leisure at the end of their six-hour workday. They filled this free time with a number of enjoyable and useful pursuits familiar to More's sixteenth-century audience. They attended lectures; they read books; they engaged in conversation, gardening, and musical performances; and they played games. Even their games were useful as well as entertaining:

> They know nothing about gambling with dice or other such foolish and ruinous games. They play two games not unlike our chess. One is a battle of numbers, in which one number plunders another. The other is a game in which the vices battle against the virtues.[1]

These games were no mere fictions. Like so many other aspects of Utopian society, they had a lengthy past in the history of medieval clerics. The "battle between virtues and vices" had been offered by Wibold, bishop of Cambrai (d. 965), to local monks as an alternative to games of dice.

The early readers of *Utopia* would not only have recognized the game of battling numbers as *rithmomachia,* also known then as the *ludus philosophorum,* or philosophers' game; if they had spent time at university, they could have played it as well. This game enjoyed a notable wave of popularity during the fifteenth and sixteenth centuries. A number of humanists, mathematicians, and educators—some well known, others less so—wrote and published rithmomachia manuals: John Sherwood, Jacques Lefèvre d'Etaples, Claude de Boissière, Benedetto Varchi and Carlo Strozzi, Francesco Barozzi, Ralph Lever and William Fulke, Gottschalk Eberbach, Abraham Riese, and August Duke of

1. Thomas More, *Utopia,* trans. H. V. S. Ogden (Northbrook, Ill.: AHM, 1949), 34–35.

Braunschweig.[2] Boards and pieces were available for purchase at the bookstores stocking these works. Manuals for two related games also appeared: the "astronomers' game," or *ludus astronomorum* (published as *Ouranomachia*), and a geometric game called *Metromachia*.

Such games may hold an intrinsic appeal to people interested in the leisure pursuits of Renaissance Europeans. Yet neither the philosophers' game nor its companions were mere pastimes for idle hours. Rithmomachia was created as an educational board game, designed to teach and exercise principles of Boethian mathematics. Its invention dates to the time when these Boethian texts were themselves being proposed as fundamental parts of a developing Latin liberal arts education, the eleventh century. Despite its many rule books from the age of print, then, it was not originally a Renaissance game at all, but a medieval one. Its playing board and pieces invite comparisons to chess, another game whose pedigree includes medieval Europe.

Chess, however, has remained with us as a modern game, while rithmomachia and its companions have faded into obscurity. Despite its flurry of publications in the middle decades of the sixteenth century, the game's popularity declined forever shortly after 1600. Even during its life span it was distinct in many ways from other games. Texts dedicated to discussing sports and pastimes generally did not mention it. Nor did it

2. Francesco Barozzi, *Il nobilissimo et antiquissimo giuoco Pythagoreo nominato rythmomachia . . .* (Venice, 1572); Claudius Buxerius (Claude de Boissière), *Nobilissimus et antiquissimus ludus Pythagoreus qui rythmomachia nominatur* (Paris, 1556); Boissière, *Le très excellent et ancien jeu Pythagorique dit rhythmomachie . . .* (Paris, 1554); Jacques Lefèvre d'Etaples, *Arithmetica decem libris demonstrata . . . Rithmimachia ludus qui et pugna numerorum appellatur* (Paris, 1496, 1507, 1514); Ralph Lever and William Fulke, *The Most Noble, Auncient, and Learned Playe, Called the Philosophers Game . . .* (London, 1563); Abraham Riese, "Arithmomachia," Dresden, Sächsische Landesbibliothek, Codex C 433, fols. 2r–21v, and "Endliche Erclerung Churfurstlicher Sexischer Arithmomachiae durch Abraham Riesen anno 1562," fols. 26r–44v; Gustavus Selenus, *Das Schach- oder König-Spiel . . . Diesem ist zu Ende angefüget ein sehr altes Spiel genandt Rythmomachia* (Leipzig, 1617); John Sherwood, *Ad reverendissimum . . . patrem . . . Marcum Cardinalem Sancti Marci vulgariter nuncupatur. Iohannis Shiruuod . . . praefatio in epitomem de ludo arithmomachie feliciter incipit* (Rome, 1482); Benedetto Varchi, letter to Luca Martini, "Trattato delle Proportioni, e Proportionalità," [Carlo Strozzi], "Dialogo sopra la particolare dichiaratione del giuoco de Pitagora," and "Regulae pro Rithmimachia Pythagorae," Florence, Biblioteca Riccardiana ms. 890, fols. 84r–130r (and several other copies). Menso Folkerts has identified another publication: Gottschalk Eberbach, n. t. (Erfurt, 1577). The sole exemplar is a manuscript copy made from the printed book: Wolfenbüttel, Herzog August Bibliothek, Codex Guelf. 238 Extrav., fols. 38r–54v. See Folkerts, "Rithmimachie," in *Mass, Zahl und Gewicht: Mathematik als Schlüssel zu Weltverständnis und Weltbeherrschung* (Weinheim: VCH, Acta Humaniora, 1989), 331–44.

find a place in the books of parlor games that began to appear in print. It made its home instead in the schoolroom, in the residences of students, and those of former students. The brief medieval manuals of its rules survive among the manuscript remains of ecclesiastical and university education. Its players could be found among those Europeans who had studied the Latin texts of Boethian arithmetic from which the game drew its basic principles.

This group, mainly clerical and male, composed only a small minority of European society. One might well question the significance of an activity that occupied only a modest part of the leisure of so few. Yet the influence of these men of learning extended, for centuries, far beyond their actual numbers; and rithmomachia reflects their learned interests as well as those of their leisure. The game's own life span encompassed something over five hundred years, an era that saw great changes in both education and society. Such a great continuity of tradition in teaching, learning, and the culture of the learned despite these changes suggests that rithmomachia's principles did indeed hold long-term significance.

Some of the claims made through the years for the value of the game found their basis in general arguments about the value of game playing or recreation in society. Other claims had broader implications; they referred to the importance of arithmetic, and of Boethian arithmetic in particular. They help to impress upon the modern reader just how important were the arguments also made by the authors of textbooks and by educational authorities about the subject's value. Arithmetic deserved study, they asserted, not simply because it served as a useful tool for the solving of computational problems. It improved the character of the person who studied it; further, it offered insight into religious truth. The philosophers' game put these principles into practice. The player benefited by coming to master the calculatory skills needed to win the game (skills that could be applied to other problems outside the game as well). More important, in contemplating and practicing the principles of arithmetic he thereby improved his soul.

To a modern reader, these are unexpected claims about a field that now seems entirely mundane and practical. They call for an examination of the source to which they refer—Boethius and his *Arithmetic*—and the attitudes about arithmetic found there. Anicius Manlius Severinus Boethius (d. 524) served as a standard textbook author for centuries of medieval education. Not only were his *Consolation of Philosophy* and other writings used for the teaching of grammar and logic, which, along with rhetoric, composed the cluster of fields in the liberal arts curriculum known as the trivium. His mathematical works, along with some spuriously attributed to him, became the basic textbooks in the other division of the curriculum known as the quadrivium, that cluster of mathematical fields composed of arithmetic, geometry, astronomy, and music.[3] Yet the significance of those quadrivial studies throughout the Middle Ages has often been less than clear.

Historians have differed greatly in their assessments about the importance—as well as the contents—of quadrivial education. To some, the quadrivium has seemed a monolithic and changeless curriculum that persisted for centuries. To them its conservative, inward-looking nature held back innovation in education and scholarship in mathematics and natural philosophy until the very advent of the scientific revolution. Others have portrayed the quadrivium as no less backward, perhaps, but not so long lived, a curriculum limited to the prescholastic era. It formed part of a system of study and classification of subjects that was swept away as early as the twelfth century by the arrival of Aristotelian logic, natural philosophy, and the scholastic disciplinary classifications used at medieval universities. More recent scholarly research, especially that on late medieval and Renaissance universities, has added far more nuance as well as substance to our understanding. This scholarship has noted that the quadrivium did survive to become part of university learning, but also underwent a number of important changes.[4]

3. See, for example, Margaret Gibson, ed., *Boethius: His Life, Thought, and Influence* (Oxford: Blackwell, 1981).

4. On the quadrivium's persistence in the medieval curriculum, see the articles by William A. Wallace, Pearl Kibre and Nancy G. Siraisi, Michael S. Mahoney, Joseph E. Brown, and John E. Murdoch and Edith Dudley Sylla in *Science in the Middle Ages,* ed. David C. Lindberg (Chicago: University of Chicago Press, 1978); Guy Beaujouan, "L'enseignement du 'Quadrivium,'" in *Par raison de nombres: L'art du calcul et savoirs scientiqua médiévaux* (London: Variorum, 1991), I; Beaujouan, "Motives and Opportunities for Science in the Medieval Universities," in *Scientific Change: Historical Studies in the Intellectual, Social, and Technical Conditions for Scientific Discovery and Technical Invention, from Antiquity to the Present,* ed. A. C. Crombie (New York: Basic Books, 1963), 219–36; Gillian Evans, "The Influence of Quadrivium Studies in the Eleventh- and Twelfth-Century Schools," *Journal of Medieval History* 1 (1975): 151–64; Evans, "Introductions to Boethius's *Arithmetica* of the Tenth to the Fourteenth Century," *History of Science* 16 (1978): 22–41; Edward Grant, "Science and the Medieval University," in *Rebirth, Reform, and Resilience: Universities in Transition, 1300–1700,* ed. James M. Kittelson and Pamela J. Transue (Columbus: Ohio State University Press, 1984), 68–102; Pearl Kibre, "The Boethian *De Institutione Arithmetica* and the Quadrivium in the Thirteenth Century University Milieu at Paris," in *Studies in Medieval Science: Alchemy, Astrology, Mathematics,*

Studies of the philosophers' game have echoed, on a smaller scale, our incomplete knowledge about the history of the quadrivium itself. The game has been the subject of several articles, appearing at widely spaced intervals, during the past century. Yet their authors have tended to approach the game almost as a fresh discovery, as a phenomenon unknown to their readers. The early history of rithmomachia and its manuscript tradition have been treated recently with great thoroughness in a German monograph by Arno Borst.[5] Yet the game's rules are still unfamiliar

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and Medicine (London: Hambledon, 1984); Kibre, "The Quadrivium in the Thirteenth Century Universities (with Special Reference to Paris)," in *Arts libéraux et philosophie au moyen âge* (Paris: Vrin, 1967), 175–91; Richard Lemay, "The Teaching of Astronomy in Medieval Universities, Principally at Paris in the Fourteenth Century," *Manuscripta* 20 (1976): 197–217, esp. 210; Michael Masi, "Arithmetic," in *The Seven Liberal Arts in the Middle Ages,* ed. David L. Wagner (Bloomington: Indiana University Press, 1983); Masi, "The Influence of Boethius' *De Arithmetica* on Late Medieval Mathematics," in *Boethius and the Liberal Arts: A Collection of Essays,* ed. Michael Masi (Bern: Peter Lang, 1981); Masi, introduction to *Boethian Number Theory: A Translation of "De Institutione Arithmetica,"* ed. and trans. Masi (Amsterdam: Rodopi, 1983); A. G. Molland, "The Geometrical Background to the 'Merton School': An Exploration into the Application of Mathematics to Natural Philosophy in the Fourteenth Century," *British Journal for the History of Science* 4 (1968): 108–25; John E. Murdoch, "Mathesis in philosophiam scholasticam introducta: The Rise and Development of the Application of Mathematics in Fourteenth Century Philosophy and Theology," in *Arts libéraux,* 215–56; Murdoch, "The Medieval Language of Proportions: Elements of the Interaction with Greek Foundations and the Development of New Mathematical Techniques," in Crombie, *Scientific Change,* 237–71; Nancy G. Siraisi, *Arts and Sciences at Padua: The Studium of Padua before 1350* (Toronto: Pontifical Institute of Mediaeval Studies, 1973); Edith Dudley Sylla, "Compounding Ratios: Bradwardine, Oresme, and the first edition of Newton's Principia," in *Transformation and Tradition in the Sciences,* ed. Everett Mendelsohn (Cambridge: Cambridge University Press, 1984), 11–43; Sylla, "The Oxford Calculatores," in *The Cambridge History of Later Medieval Philosophy,* ed. Norman Kretzmann et al. (Cambridge: Cambridge University Press, 1982), 540–63; Sylla, "Science for Undergraduates in Medieval Universities," in *Science and Technology in Medieval Society,* ed. Pamela O. Long (New York: New York Academy of Sciences, 1985); James Weisheipl, "The Place of the Liberal Arts in the University Curriculum during the XIVth and XVth Centuries," in *Arts libéraux,* 209–15; Alison White, "Boethius in the Medieval Quadrivium," in Gibson, *Boethius,* 162–205.

5. Modern scholarship on rithmomachia has included R. Peiper, "Fortolfi Rythmimachia," *Zeitschrift für Mathematik und Physik: Supplement zur historisch-literarischen Abtheilung; Abhandlungen zur Geschichte der Mathematik,* vol. 3 (Leipzig: Teubner, 1880), 167–227; E. Wappler, "Bemerkungen zur Rhythmomachie," *Zeitschrift für Mathematik und Physik, Historisch-literarische Abtheilung* 37 (Leipzig: Teubner, 1892), 1–17; David Eugene Smith and Clara C. Eaton, "Rithmomachia, the Great Medieval Number Game," *American Mathematical Monthly* 18 (1911): 73–80; John F. C. Richards, "Boissière's Pythagorean Game," *Scripta Mathematica* 12 (1946): 177–217; Richards, "A New Manuscript of a Rithmomachia," *Scripta Mathematica* 9 (1943): 87–99, 169–83, 256–64; Beaujouan, "L'enseignement du 'Quadrivium'"; Wolfgang Breidert, "Rhythmomachie und Globusspiel," *Mitteilungen und Forschungsbeiträge der Cusanus-Gesellschaft* 10 (1973):

enough to merit describing before turning to its context and significance. Doing so requires a survey in turn of the Boethian texts on which they are based.

Boethius's *Arithmetic* and *Music* are the surviving remains of his larger, unfinished project of organizing a Latin liberal arts curriculum. The *Arithmetic* is essentially a loose translation of writings by the second-century scholar Nicomachus of Gerasa. Much of the *Music* is probably also based on a text of Nicomachus now lost, with additional material from Ptolemy's *Harmonics* and other works.[6] Nicomachus, and Boethius in turn, participated in the revival of Pythagorean thought in later antiquity. Pythagorean thought had received a wide early audience in Plato's *Timaeus*. It was the one Platonic dialogue that was translated into Latin (in part) with a commentary by Chalcidius in the fourth century A.D. These texts would add to the perceived importance of Boethius for medieval Latin readers.

From these sources and a few others, they were taught to think of the creation of the world as the imposition of order upon formless matter. That order was numeric in nature and generated from the smallest units, the numbers one through four. In the *Timaeus* they were described most clearly in Plato's discussion of the World Soul, composed of the first three powers of two and three, in the double series 1:2:4:8 and 1:3:9:27.[7] Boethius referred to these notions briefly in the *Arithmetic* and developed them in more detail in the *Music*. These same proportions can be seen in the distance of the planets from one another, the combination of elements as they form the substances of the physical world, and the organization of the human soul, which itself resembles the World Soul in miniature.

155–71; Gillian Evans, "The Rithmomachia: A Mediaeval Mathematical Teaching Aid?" *Janus* 63 (1976): 257–73; Adriano Chicco, "La Rithmomachia," in *Bonus Socius: Biudragen tot de cultuurgeschiedenis van het schaakspel en andere bordspelen* (The Hague: Koninklijke Bibliotheek, 1977), 81–101; Detlef Illmer et al., *Rythmomachia: Ein uraltes Spiel neulich entdeckt* (Munich: Hugendubel, 1987). See also Grant, "Science and the Medieval University"; Adriano Chicco and Giorgio Porreca, *Dizionario enciclopedico degli scacchi* (Milan: U. Mursia, 1971). The most significant scholarship by far has been that of Arno Borst, *Das mittelalterliche Zahlenkampfspiel* (Heidelberg: Carl Winter Universitätsverlag, 1986); Borst, "Science and Games," in *Medieval Worlds: Barbarians, Heretics, and Artists in the Middle Ages,* trans. Eric Hansen (Chicago: University of Chicago Press, 1992), 195–212.

6. Calvin Bower, introduction to Boethius, *Fundamentals of Music,* trans. Bower (New Haven: Yale University Press, 1989), xxvi–xxix; John Caldwell, "The *De Institutione Arithmetica* and the *De Institutione Musica,*" in Gibson, *Boethius,* 135–54.

7. *Timaeus* 34c–36d.

Much of the contents of both Boethian textbooks consists of definitions of types of number and the relationships between them, and discussions of their properties. Significant types include odd and even, prime and composite, perfect numbers, and their products. Relations between numbers (proportions) may be based on equality or inequality. The latter have several types: multiplex (of the form x:1), superparticular [$(x + 1)$:x], and superpartient [$(x + 2)$:x, $(x + 3)$:x, and so on].[8] These proportions can also have a mean established between them. Boethius identified three means. In the first type, arithmetic, the mean falls at the arithmetic midpoint between the two extremes (mean, or $m = (a + b) \div 2$, for example, 1:2:3). A geometric mean produces similar ratio with the two extremes: (a:m = m:b, for example, 1:2:4). The harmonic mean creates the most consonant proportions ($m = 2ab \div (a + b)$, for example, 3:4:6). Particular subsets of these proportions constitute musical consonances: multiple and superparticular proportions using the numbers one through four. These terms and proportions are useful for solving some kinds of calculatory problems, such as the construction of a musical scale. Nonetheless, Boethius claims that their ultimate value lies in their ability (through such applications or on their own) to turn the mind of the scholar toward an understanding of the divine.

Boethius's *Arithmetic* is more difficult to follow for a modern reader than many other ancient mathematical texts, for example, Euclid's *Elements of Geometry.* Euclid's text proceeds by means of proofs and demonstrations; it sets problems and solves them, and then often uses those solutions in subsequent problems in turn. These sorts of topics and methods are still part of modern expectations about mathematical subjects and how to study them. Indeed, the teaching of geometry is still based to some degree on Euclid's text. Boethius's *Arithmetic,* on the other hand, seems to devote far too much attention to defining terms that seem arbitrary at best, and far too little to calculation, proof, or problem solving. Boethius's approach to mathematics also differs from the approaches taken by most of the mathematical works introduced into European society from the twelfth century on via contact with the Islamic world. Like Euclid's geometry, they seem a better match for modern expectations

8. On the importance to medieval mathematicians of the Boethian practice of "denominating" or naming and classifying these intervals, even when doing so interfered with or distorted the understanding of non-Boethian mathematical thinkers such as Euclid or Eudoxus, see Michael S. Mahoney, "Mathematics," in Lindberg, *Science in the Middle Ages,* 163–64.

about both the style and the substance of mathematics texts: they pose problems both practical and abstract and then solve them, proceeding from simple principles to more complex ones.

So too the vocabulary of Boethian studies differs somewhat from modern terminology. Modern technical usage distinguishes between the terms *ratio* and *proportion*. The latter refers to a relationship of equality between ratios,[9] though in common usage the two terms are often used less precisely as synonyms. Boethius did not maintain the modern technical distinction in Latin. He used the term *proportio* far more often than *ratio* to indicate a relationship between two numbers.[10] While Euclid made this distinction (as in *Elements,* books 5 and 8), not all medieval scholars of Euclid maintained it. A broad rather than narrow usage of the term *proportio*—that is, using it simply to refer to a relationship between two numbers—became common in medieval Latin scholarship.[11]

By the Renaissance this terminology had begun to move into vernacular languages as well, although it eventually encountered resistance from humanists. The mathematician Francesco Barozzi (1537–1604), for example, cited the use of the neologism *proportionalitas* as one of the many errors and barbarisms found in the astronomy textbook of the medieval scholar Giovanni da Sacrobosco (1195–1256). Barozzi blamed Cicero as the source of the error for translating two different Greek terms as *proportio* (he noted that *ratio* should be the other term). The blunder, he continued, was then perpetuated by Boethius, later by Campanus of Novara (1220–1296), and others.[12] Nonetheless, the vernacular term *proportion* remained in general use and is the usage often employed in some modern fields such as the history of art. It has thus seemed anachronistic to introduce into the present study a distinction that was not present in the sources. Therefore *proportion* is used throughout the present study in the broad and general sense in which it appears in the historical texts.

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9. As, for example, $a/b = c/d$.

10. See entries under *proportio, ratio* in Michael Bernhard, *Wortkonkordanz zu Anicius Manlius Severinus Boethius De institutione musica* (Munich: Verlag der Bayerischen Akademie der Wissenschaften, 1979); Masi, "Boethian Number Theory and Music," in *Boethian Number Theory,* 26n.

11. Murdoch, "Medieval Language of Proportions"; J. D. North, *Richard of Wallingford, an Edition of His Writings with Introductions, English Translation, and Commentary* (Oxford: Clarendon, 1976), 2:55–57; see also *The Thirteen Books of Euclid's Elements,* trans. Thomas Heath, 2d ed. (New York: Dover, 1956), 2.112–86.

12. Francesco Barozzi, letter to P. Cristoforo Clavio, Milan, Codex Amb. S. 81, fol. 259, in B. Boncompagni, "Intorno alla vita de ai lavori di Francesco Barozzi," *Bullettino di bibliografia di storia delle scienze matematiche e fisiche* 17 (1884): 835.

The pervasiveness of this Boethian terminology in so many different fields—including those of modern scholarship—suggests that despite many idiosyncrasies from the point of view of more modern mathematical studies, Boethius's quadrivial writings proved to be a useful part of the curriculum. They were useful, in fact, in several different ways. In his authorship of such a wide range of textbooks (including his *Music* and *Arithmetic*), Boethius was able to assert the connectedness of their subjects. He emphasized the unity among such diverse topics as theology, natural philosophy, and arithmetic. This Platonic sense of the unity of learning took root sufficiently to survive first the advent and then the dominance of Aristotelianism in later medieval scholarship. Boethius's descriptions of proportions and means could even be applied productively to some of these new scholastic subjects, such as the study of bodies in motion.

Quadrivial studies also had some direct practical applications whose usefulness continued long after the rise of university education. These applications included computus (calendar reckoning, especially for religious holidays) and chant singing (producing a correct musical scale for the singing of religious chant). These applications, based as they were in religious observance, were also consistent with a model of education that understood the spiritual and moral development of the student as its highest goal. These devotional and liturgical practices extended the quadrivium and its philosophical foundations beyond texts into the more general realm of learned culture. The singing of chant, for example, could serve to remind educated singers and listeners alike of the close connections among musical intervals, the order found in the divinely created world, and the religious observance being undertaken in the act of singing. In its own way, rithmomachia moved these same Boethian principles from texts into another arena of activity.

Rithmomachia's rules of play are not altogether different in their basic outlines from a number of board games. The game employs two players, who alternate moves of a single piece per turn; a checkered board like a lengthened chess board; and forty-eight playing pieces, each inscribed with numbers.[13] Its precise rules always varied a bit from one game manual to the next, and several manuals took note of such variations. The earliest manuals did not even specify a precise size for the gaming board, though they suggested a minimum of eight squares wide

13. Evans, "The Rithmomachia," 264–65.

and twelve long. By the twelfth century, many sources specified eight by sixteen, the size of two chessboards end to end. The playing pieces on each side are divided into three ranks; one piece in the last rank is replaced by a stepped pyramid. Eventually the two teams came to be distinguished by color, and the three ranks of pieces by shape: circles, triangles, and squares. The first rank (circles) generally could move to the next space, the other ranks two and three squares. Some game manuals noted that players might decide before beginning to play whether to allow diagonal moves.

Rules for the capture of enemy pieces and the establishment of victory all depend upon the numbers written on the pieces. These numbers exemplify the basic types of Boethian proportions. The two opposing sides are distinguished as Odd and Even, and the first four odd and even numbers are those that appear on the first four circles of each side: 3, 5, 7, 9 (Boethius defines 1 as the unit and not as a number) and 2, 4, 6, 8. The second row of circles is produced by squaring each number of the first row. Thus each pair in the first and second rows illustrates the multiplex proportion (that is, x:1). So for the odd side the second row is 9, 25, 49, 81. The ratio 9:3 reduces to 3:1, which in Boethian terms is called the "triple multiplex proportion"; 25:5 equals 5:1, 49:7 equals 7:1, and so on.

Evens				Odds			
2	4	6	8	3	5	7	9
4	16	36	64	9	25	49	81

The next row, the first of the triangles, produces a series of superparticular proportions [$(x + 1):x$] with the second row. The numbers are generated by adding the numbers of the first two rows.

Evens				Odds			
2	4	6	8	3	5	7	9
4	16	36	64	9	25	49	81
6	20	42	72	12	30	56	90

Comparing the third row with the second produces 6:4 (= 3:2), 20:16 (= 5:4), and so on; this reduces to the series of superparticulars 3:2, 5:4, 7:6, 9:8 and 4:3, 6:5, 8:7, 10:9. The first term of each superparticular type is added to the relevant number in the third row to produce the fourth row.

Evens				Odds			
2	4	6	8	3	5	7	9
4	16	36	64	9	25	49	81
6	20	42	72	12	30	56	90
9	25	49	81	16	36	64	100

This produces another series of superparticulars when comparing the numbers of the fourth row to those of the third: 9:6 (= 3:2), 25:20 (= 5:4), and so on. Once again they reduce to 3:2, 5:4, 7:6, and 9:8 for the evens; and 4:3, 6:5, 8:7, 10:9 for the odds.

The rows of squares illustrate superpartient proportions ($x + 2{:}x$, $x + 3{:}x$, . . .). As in the earlier rows, the numbers are the sum of those in the two previous rows.

Evens				Odds			
2	4	6	8	3	5	7	9
4	16	36	64	9	25	49	81
6	20	42	72	12	30	56	90
9	25	49	81	16	36	64	100
15	45	91	153	28	66	120	190

By this point the proportion types—and their names—have become more complex. The ratio 15:9 (= 5:3) is called a *superbipartiens tertias;* that is, the first number exceeds the second by 2, and the second number is 3. The series is thus: 5:3, 9:5, 13:7, 17:9; and 7:4, 11:6, 15:8, 19:10. These series are maintained in the fourth row.

Evens				Odds			
2	4	6	8	3	5	7	9
4	16	36	64	9	25	49	81
6	20	42	72	12	30	56	90
9	25	49	81	16	36	64	100
15	45	91	153	28	66	120	190
25	81	169	289	49	121	225	361

The pyramids are stepped and have numbers on each step. The pyramid of the even side has six steps; its base is marked 6 on one side and its square, 36 on another, as befits the square shape of each level. The next level is marked 5 and 25, and so on. The top is marked with the sum of the squares on the steps, or 91 (36 + 25 + 16 + 9 + 4 + 1); it replaces the square piece "91" in the chart. The odd side's pyramid begins with larger numbers, 8 and 64; it is missing the top three layers (and is often called *tricurta*), so that the highest step is 16 and 4. Its top number is 190, and it replaces that square in its team's squadron. The pieces begin play with the round ranks front and center, with the others fanning out behind and to the sides.

The numbers on the pieces themselves thus serve as examples of the proportional types that are the subject of so much description and discussion in Boethius's *Arithmetic* and *Music*. Simply contemplating the board, without even beginning to play, would serve to rehearse these numerical relationships in the players' minds. Early versions of the game naturally employed the Roman numerals that were still in common use, rather than modern Hindu-Arabic numbers. The awkwardness of computing with Roman numerals meant that those who used them needed to rely heavily on memorization, as well as making use of computational tools like the abacus. For the users of Roman numerals, the game itself was also valuable as a memory aid.[14]

The rules of play enjoyed a number of variations. Indeed, some sources suggested that players might agree to specific rules before commencing, thus allowing the game to be played at a range of skill levels. In general, however, playing pieces may capture one another by several means. A piece that moves to a position from which it may then occupy the same space as the opponent's piece bearing the same number captures that opponent's piece. Two pieces may take a third by addition if they could both move to occupy the opponent's space and their sum equals the opponent's number. The same may be done by multiplication, and Fulke and Lever also allow for subtraction or division. One side may "besiege" an opponent's piece and capture it simply by moving pieces (with whatever numbers) so that they occupy all the spaces into which that piece could move. The pyramids are vulnerable to capture not only by means of the numbers on their top but any on their sides. According

14. One early version employed Greek notation: Oxford, St. John's College ms. 17, fol. 56v. See Evans, "The Rithmomachia," 266.

to some manuals, however, a pyramid may be ransomed by substituting the capture of the single piece matching the vulnerable number.

The establishment of victory constitutes a second phase of the game. Here the player, using his own pieces (and in some versions the captured pieces of the opponent as well, if desired) assembles in a line Boethian proportions with their means. Once the player has announced this intent, his opponent may no longer capture pieces. The highest level victory involves setting out arithmetic, geometric, and harmonic means. This detailed endgame emphasizes the complexity of the game's rules.

A real barrier to the game's enjoyment by modern players is that the numbers can seem so arbitrary. Yet the game's evident appeal for so many years even outside the classroom suggests that its medieval and Renaissance players were not so troubled. Game descriptions might note that these numerical relationships are complex, but did not suggest that players found them arbitrary or irrelevant. That difference of sensibility results from the presence or absence of their source, Boethius, in one's educational background. Rithmomachia was played as long as Boethius's *Arithmetic* was taught; the game and the curriculum disappeared together. In order to understand the long-term success of rithmomachia, we need to follow the curricular trail of the Boethian texts that made these numbers relevant to the game's players. The use of the Boethian texts in medieval education is itself documented only very incompletely. Because of the close relationship between the game and Boethius's *Arithmetic,* rithmomachia's presence helps indicate the presence of that text in turn. It helps to fill in our incomplete knowledge about mathematics instruction in general and quadrivial studies in particular. In some ways then, the present study is at times as much about the history and significance of the teaching of Boethius's *Arithmetic* as it is about the game itself.

This linked history of textbook and game also helps to highlight the fact that the traditions of medieval and Renaissance European mathematical education were not one but several. These traditions developed differently in different contexts, be those contexts intellectual or institutional and social. For neither Boethius nor rithmomachia could be found everywhere that Europeans gathered in classrooms to study numbers and computation. Part of this variation was regional. Interest in rithmomachia varied greatly across Europe, according to the evidence of manuscript survivals and textual references. The game was best known in the north (particularly in England and France), but was not found in Italy until the fifteenth century and seems to have been completely unknown on the Iberian peninsula. Italian schooling in arithmetic and mathematics stands out in particular as distinctly different from the curriculum of northern universities.

To emphasize this regional variation may seem to restrict still more the world of Boethius's textbooks and rithmomachia to a small circle within European society. Yet it also emphasizes the role of Boethius's teachings. They were important not just as a body of information transmitted in a cluster of specialized texts read by a very few scholars and teachers, but as a curriculum followed by a significantly larger group of people. As a course of study it existed in particular social settings with particular pedagogic goals. Most of the persons who came into contact with it did so as students, not as instructors or even as future scholars. These students carried the principles of the quadrivium along with those of the rest of their education, mastered well or poorly, away from their student days and into the rest of their lives. Quadrivial study did not proffer the image of some sort of professional mathematician or even a keeper of quantified records as its ideal product. Rather, it tried to produce people of virtue, men (since most of these educational institutions were exclusively male) who understood the basic features of the ordering of the world and strove to build an inner character that reflected such order. These features of cosmic order were eternally and changelessly true and divine in origin. Thus the lack of change in textbooks and content was an explicit part of the field's identity and claims to importance and validity. Rithmomachia aided those goals and survived as long as teachers of mathematical subjects understood their task in those terms.

Modern mathematics education has kept only the barest shadow of these goals, found mainly in its own claims to foster disciplined and organized thought in its students. The task of character development now falls mainly to the humanities, a cluster of subjects that hardly existed as such in the early days of the university. This change altered not only the nature of mathematical instruction and study, but also the field's relationship to other fields. Modern classifications of knowledge would see mathematics as related mainly to scientific subjects and would evaluate the success or failure of such disciplines mainly in terms of the progressive accumulation of knowledge and advancement of theories. Quadrivial studies, on the other hand, claimed explicitly to be related to a much broader range of learned activities and pursuits, in which notions of scientific progress might or might not hold much meaning. These educational goals help to account for a feature of medieval mathematics (and especially of quadrivial studies) that has long vexed modern scholars, its apparent changelessness and lack of progress.

Yet a good deal of evidence suggests that interest in the game and Boethian mathematics was not entirely restricted to the level of general education; it could be found among specialists as well. Among them were those medieval scholars known as *calculatores,* who were interested in mathematical physics. The discussions in game manuals of rithmomachia's value in improving not just the character but also the mood of the player help to highlight the connections between Boethian mathematics, medicine, and subjects that would now be called psychology. This link is especially visible in the references to melancholy, a condition to which scholars were considered prone and which some writers claimed the game could help dissipate.

From the later Middle Ages through the Renaissance rithmomachia also held an attraction for students of astronomy and astrology, fields closely related in turn to natural magic. Several of the authors of Renaissance game manuals showed an interest, in their activities or publications, in these subjects. Rithmomachia itself was never classified as magic, nor did it fall victim to any of the controversy attached to these studies. They shared a relationship to the Boethian tradition, however, which helped provide the theories of cosmic harmony that underlay them. The game's presence in these circles helps to show how fully the principles behind much natural magic were integrated into both curriculum and practice at universities, principles to which both specialists and general students were exposed.

Rithmomachia's status as a leisure activity of learned clerics also helped to identify them as a cultural group: as an elite with shared education and values, as well as shared pastimes. Those values included a close identification between mathematics or science on the one hand, and theological study or religious contemplation on the other. This association, like the Boethian textbooks themselves, survived the rise of university education and became a part of it. Later, this association found a particular resonance with many northern humanists. In so doing it served as a point of intellectual and cultural continuity from the prescholastic culture of the eleventh century through the Renaissance.

Ralph Lever and William Fulke's game manual, *The Most Noble, Auncient, and Learned Playe, Called the Philosophers Game,* offers a fuller understanding of the game's rules, strategies, and variations than could any modern description. This manual and that of Boissière were the most thorough and complete of the published game manuals of the Renaissance. Their decision to publish the work in English made it the natural choice for inclusion here. While the number of variations in rules ensures that no single version can truly be seen as definitive, theirs represents the game as known in a time and place in which it was perhaps in greatest use. The philosophers' game was an activity, not a text, and so no text can reproduce fully the immediacy of the experience of play. Still, Lever and Fulke's manual offers modern readers a closer approach to that experience.

The history of the philosophers' game offers more than an understanding of the leisure culture of learned elites. It also helps point the way toward understanding the roles and importance of the Boethian tradition of instruction in arithmetic and its mathematical philosophy. By examining the geographic and cultural distribution of the philosophers' game, its players, and the contexts in which it belonged, we may hope to find a means "utilis et jocundus" (in the words of an eleventh-century treatise) both to follow the Boethian mathematical tradition and to assess its significance as part of medieval and Renaissance thought and culture. The game's position at the boundary between learning and leisure, text and act, and especially mathematics and morality, provides a rare opportunity to examine those relationships in European learned culture.