Chapter 10
Classical Dynamic Models

Classical dynamic models are those which are appropriate for describing the behavior of deterministic dynamic systems. The best-known examples of these are the systems examined in celestial mechanics and those related to electromagnetic phenomena. Typically, events of this sort are described by systems of second order ordinary or partial differential equations.[28] Chemical systems, on the other hand, can be frequently described by first order ordinary differential equations, since in these systems the reaction rates (rates of change of concentrations) are directly related to the concentrations of the various reacting substances and of the catalysts. All systems so described are typically deterministic in the sense that a set of initial conditions and the parameters of the equations determine the fate of such a system for all time.

In behavioral science, the present trend is away from deterministic models. The trend is well justified in view of the recognition that it is all but impossible to specify initial conditions and the laws of behavior so exactly as to predict the behavior of a living system for any significant length of time. Stochastic models have been put forward as more appropriate in behavioral contexts. Nevertheless, much can be learned by casting a behavioral situation into a deterministic model. This is done not so much with the view of finding deterministic models adequate for the behavioral systems to be described as for the purpose of revealing some general features of the situation.

Nevertheless, the predictive potential of deterministic models is not always to be discounted in behavioral science. It is sometimes considerable if the behaving systems are large enough. For example, the time course of C averaged in a very large population sample playing a Prisoner's Dilemma game may very well be "determined" in the sense that it could be accurately replicated by another large sample from the same population playing the same game. There is thus a curve describing the time course and therefore a set of dynamic laws from which the equation of the curve can be derived. What one must guard against is jumping to the conclusion that one has discovered *the* dynamic laws once a successful derivation has been made. A corroboration of a model never proves that the model represents reality but only that it can be *taken* to represent the phenomenon studied.

In the models to be presently constructed, we shall not seek an adequate representation of our data, let alone definitive dynamic laws. We merely wish to investigate the consequences of certain assumptions concerning a *possible* dynamic underlying the Prisoner's Dilemma game.

Classical dynamic models are usually represented in the form of systems of differential equations, in which the independent variable is time and the dependent variables are the quantities whose time courses determine the sequence of states through which the system in question passes. Our independent variable will also be time, which is here assumed to be measured by the number of elapsed plays of a game (or games). For our dependent variables we can take either the unconditional propensities for cooperative response, that is C_1 and C_2, or the conditional propensities, x_i, y_i, z_i, w_i. In the former case, we shall speak of first order dynamic models; in the latter case, of second order dynamic models. These models will be either dynamic extensions of the equilibrium models examined in Chapter 8 or deterministic versions of stochastic models examined in Chapter 9.

Recall that in the equilibrium model each subject was assumed to adjust some variable so as to maximize his expected gain. When the adjusted variables are C_1 and C_2, the model leads to the counter-intuitive "strategic" solution of Prisoner's Dilemma, namely to eventual one hundred percent noncooperation (cf. p. 130). We therefore already know where a dynamic extension of this model will lead. We shall, accordingly, bypass this version and pass directly to a deterministic version of the stochastic model. In the stochastic model C_1 and C_2 undergo modifications which result not from attempts to maximize expected gains but rather from immediate reactions to the outcomes. Thus C_1 suffers a positive increment following the outcome C_1C_2, etc. In the classical version the same thing happens. But now the process is described by a deterministic system of differential equations, namely

$$dC_1/dt = \alpha_1 C_1 C_2 - \beta_1 C_1(1 - C_2) - \gamma_1 C_2(1 - C_1) + \delta_1(1 - C_1)(1 - C_2), \quad (94)$$

$$dC_2/dt = \alpha_2 C_1 C_2 - \beta_2 C_2(1 - C_1) - \gamma_2 C_1(1 - C_2) + \delta_2(1 - C_2)(1 - C_1), \quad (95)$$

where α_i, β_i, and γ_i are positive constants. The sign of δ_i, however, is ambivalent.

The terms on the right-hand side of (94) and (95) are interpreted as follows. The term involving α_i represents the positive contribution to dC_1/dt due to a double-cooperative outcome. The term involving β_i represents the negative contribution due to an unreciprocated cooperation. The terms involving γ_i represent the negative contribution due to successful defection. As for δ_i, we are not sure of the sign of its contribution. On the one hand, the punished defection may contribute to an increase in the probability of cooperative choices; but on the other hand, repeated double defections may contribute to increased distrust and so may have the opposite effect. Accordingly, the sign of δ_i will remain ambivalent.[29]

Systems like (94) and (95) are frequently studied by means of the so-called phase space. In our case, the coordinates of this space are C_1 and C_2. At each point (C_1,C_2) the magnitudes of dC_1/dt and of dC_2/dt are represented by a vector. One can imagine a particle moving with a velocity whose horizontal and vertical components are represented by the components of that vector, namely dC_1/dt and dC_2/dt. The motion of the particle reflects the time courses of C_1 and C_2 and hence the dynamics of the system.

Let us arbitrarily set $\alpha_i = \gamma_i = 1$; $\beta_i = 2$; $\delta_i = 1$. This reduces Equations (94) and (95) to

$$dC_1/dt = 5C_1C_2 - 3C_1 - 2C_2 + 1, \tag{96}$$

$$dC_2/dt = 5C_1C_2 - 2C_1 - 3C_2 + 1. \tag{97}$$

It is apparent that there are two equilibrium points. In fact, these can be obtained directly by setting the right-hand side of Equations (96) and (97) equal to zero and solving for C_1 and C_2.[30] The equilibria turn out to be at

$$C_1 = C_2 = \frac{5+\sqrt{5}}{10} \text{ and at } C_1 = C_2 = \frac{5-\sqrt{5}}{10}. \tag{98}$$

Moreover, it is easily inferred from the appearance of the phase space that the lower equilibrium is stable while the upper one is unstable. Once C_1 and C_2 become sufficiently large, they are driven into the upper right-hand corner of the phase space where $C_1 = C_2 = 1$. Otherwise the C's are driven toward the stable equilibrium point. In other words, this model (with the values of parameters as chosen) predicts that if the players cooperate sufficiently frequently initially, they will eventually lock-in on cooperation. If the initial frequencies of cooperative choices are not large enough, the players will still show a residual amount of cooperation (at the lower equilibrium). The reason for the latter result is not far to seek. Our assumption is that $\delta > 0$ keeps "scaring" the players away from *DD*.

The picture looks different if $\delta < 0$. We now arbitrarily set $\alpha_i = 4$; $\beta_i = 3$; $\gamma_i = 2$; $\delta_i = -1$. Equations (94) and (95) now become

$$dC_1/dt = 8C_1C_2 - 2C_1 - C_2 - 1, \qquad (99)$$

$$dC_2/dt = 8C_1C_2 - C_1 - 2C_2 - 1. \qquad (100)$$

Now there is only one equilibrium in the region bounded by the unit square, which is the region where C_1 and C_2 have meanings as probabilities. From the appearance of the phase space, we see that here the all-or-nothing Richardson Effect is operating: C_1 and C_2 will be either driven toward 1 or toward 0.

In general if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$ equilibria will lie on the line $C_1 = C_2$. Let us see how their nature depends on the parameters. We obtain the equilibria by setting dC_1/dt and dC_2/dt equal to zero; this reduces the right-hand side of (94) to

$$F(C) \equiv (\alpha + \beta + \gamma + \delta)C^2 - (\beta + \gamma + 2\delta)C + \delta. \qquad (101)$$

The quadratic polynomial has real roots if

$$(\beta + \gamma + 2\delta)^2 \geq 4\delta(\alpha + \beta + \gamma + \delta), \qquad (102)$$

which upon expansion and simplification, reduces to

$$(\beta + \gamma)^2 \geq 4\alpha\delta. \qquad (103)$$

In summary, we have the following results:

(1) If $\delta > 0$ and if $(\beta + \gamma)^2 < 4\alpha\delta$, the system will be driven unconditionally toward full cooperation;

(2) If $\delta > 0$ and $(\beta + \gamma)^2 > 4\alpha\delta$, then, depending on the initial value of C, the system will be either driven toward complete cooperation or will be stabilized at a certain positive value of $C_1 = C_2$;

(3) If $\delta < 0$, then, depending on the initial value of C, the system will be driven either toward full cooperation or toward full defection.[31]

The method can, of course, be extended to the general case where the parameters of the two players are not the same, but we shall not pursue this generalization here.

A Simplest Second Order Model

By a second order model we mean one where the principal variables are not the probabilities of cooperative choice but the *contingent* probabilities, such as x, y, z, and w. We shall consider the simplest case where $y_i = z_i = 0$; $w_i = 1$ $(i = 1, 2)$, so that only x_1 and x_2 are variable. This is the case of the two tempted simpletons (cf. p. 73). We have already examined the statics of that situation. Now let us consider the dynamics.

Suppose the simpletons adjust their x's at a rate proportional to the *gradient* of the expected gain. In other words, if increasing x_1 results in a large positive change in the expected gain of player 1, then he will increase x_1 rapidly; if x_1 results in a small negative change in the expected gain, he will decrease x_1 slowly, etc.

Formally, we represent this by the following equations:

$$\frac{dx_1}{dt} = k_1 \frac{\partial G_1}{\partial x_1}, \qquad (104)$$

$$\frac{dx_2}{dt} = k_2 \frac{\partial G_2}{\partial x_2}, \qquad (105)$$

where k_1 and k_2 are constants. Substituting for $\partial G_1/\partial x_1$ and $\partial G_2/\partial x_2$ the expressions derived earlier (cf. p. 132), we have, assuming $P = -R$,

$$\frac{dx_1}{dt} = k_1 \frac{x_2^2(3T + R) + 2x_2(R - T) - 2T}{(2 + x_1 + x_2 - 3x_1x_2)^2}, \qquad (106)$$

$$\frac{dx_2}{dt} = k_2 \frac{x_1^2(3T + R) + 2x_1(R - T) - 2T}{(2 + x_1 + x_2 - 3x_1x_2)^2}. \qquad (107)$$

Equations (106) and (107) constitute the dynamics of the system.

The numerators of the right sides of Equations (106) and (107) represent the same parabola, and the denominators are always positive. Hence the signs of dx_1/dt and dx_2/dt are determined entirely by the signs of the numerators. If $T < 3R$ the parabola has a single root in the unit interval, namely (cf. p. 133)

$$x = \frac{T - R + \sqrt{R^2 + 7T^2}}{3T + R} = x^*. \tag{108}$$

Therefore the unit square is divided into four quadrants, in which the signs of the derivatives are as follows:

where $x_1 > x^*$; $x_2 > x^*$, $dx_1/dt > 0$; $dx_2/dt > 0$;
where $x_1 > x^*$; $x_2 < x^*$, $dx_1/dt < 0$; $dx_2/dt > 0$;
where $x_1 < x^*$; $x_2 > x^*$, $dx_1/dt > 0$; $dx_2/dt < 0$;
where $x_1 < x^*$; $x_2 < x^*$, $dx_1/dt < 0$; $dx_2/dt < 0$.

Figure 28 shows a schematic representation of this dynamic system.

Once the motion of a point in (x_1,x_2) space is determined, the motion of a point in (C_1,C_2) space is also determined, since every pair of values (x_1,x_2) "maps" upon a pair of values (C_1,C_2) according to Equations (24) and (25). The model is deterministic with respect to the motion of a point in (C_1,C_2) space. However, the position of a point (C_1,C_2) does not determine a particular outcome, since in the present context C_1 and C_2 are only probabilities of choosing C. At most a pair of probabilities (C_1,C_2) determine the corresponding frequencies in a population of "identical players." Recall, however, that we have assumed that x_1 and x_2 are adjusted with reference to corresponding expected payoffs, i.e., with reference to C_1 and C_2 viewed as probabilities.

How these probabilities are *instantaneously* estimated by the players is an embarrassing question for the theory.

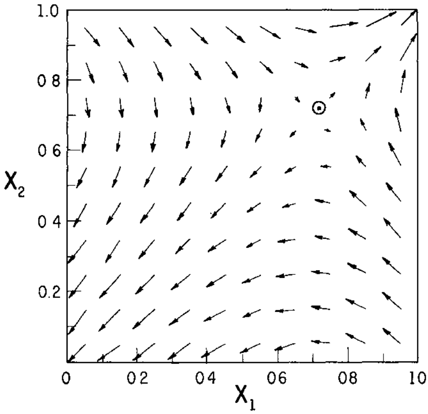

Figure 28. The dynamics of playing Prisoner's Dilemma by adjusting x so as to maximize expected gain. For purely esthetic reasons the game represented here was chosen as the limiting case where $R = T$. Also $P = -R$ and $S = -T$. The fixed parameters are $y = z = 0$; $w = 1$.

The circled point represents the (unstable) equilibrium at $x_1 = x_2 = x^* = \sqrt{2}/2$, which obtains in this case. If the initial values of x_1 and x_2 are the coordinates of the tail of any of the arrows, the direction of the arrow represents the direction of motion of the point (x_1, x_2) while the length of the arrow represents the speed of the motion. Thus the line inclined 135° to the horizontal drawn through the equilibrium represents a "watershed." All the motion above and to the right of this watershed will be toward the upper right-hand corner ($x_1 = x_2 = 1$), and all the motion below and to the left will be toward the lower left-hand corner ($x_1 = x_2 = 0$).

After all, the players do not have access to a "population of identical players," in which the probabilities can be observed as frequencies.

The only possible way in which the adjustment model can be realized in practice is for the players to make their adjustment extremely slowly, waiting until the values accruing to them become good estimates of their expected payoffs before making the next adjustment. This involves waiting until each new equilibrium is established (i.e., the steady state determined by the Markov chain involving the newly adjusted value of x) and then waiting until the steady state has been maintained long enough to estimate the new expected payoff. During all this time, moreover, the player must remember the average payoff determined by the value of x used previously, in order to compare it with the present value, so as to know in which direction the next adjustment should be made. In the meantime the other player must bide his time until it is his turn to go through this process.[32]

When we recall that all this insures the adjustment of only one pair of parameters, not four pairs contained in the general model of this type, we see even more clearly the tremendous gap that separates a formal model from what can be reasonably expected in "real life," and "real life" in this case denotes only the laboratory situation, not real "real life."

The situation would be utterly discouraging if one expected the models to be faithful replicas of behavior. The purpose of all models, however, is, or ought to be, quite different. Every model can be expected to represent reality only in an "as if" fashion. The goal of the theoretician working with mathematical models in behavioral science ought to be not detailed prediction but an understanding of what is involved in the phenomenon in question. An understanding is achieved by examining the salient features of the models, then searching for these features (or, perhaps, noting their absence) in what is observed. Thus, a vitally important feature of the model in which state-conditioned propensities are adjusted is the Richardson Effect. The lock-in phenomenon observed in the data seems to be also a manifestation of a similar effect. The problem before us now is to demonstrate the reality of the effect (or to refute it) under various conditions, to search for its psychological underpinnings, etc. The various models serve here as ways of viewing the behavioral situation. A good model is one which suggests ways of bringing out important features of the situation not hitherto considered or understood. Every mathematical model in behavioral science should serve as a point of departure for investigations, not as a conclusive formulation of a theory.

Summary of Part II

We have proposed a number of models, some stochastic, some deterministic of the sort of process which may be governed by the interactions of two players playing Prisoner's Dilemma. The usefulness of the models in constructing a testable theory of the process is severely limited by the quickly increasing number of parameters which must be estimated in order to compare the predictions of the models with empirical results. Nevertheless, an important feature of the experimental results already emerges in some of the models, namely the lock-in effect which drives the players to either of the extremes. This lock-in effect is explicitly built into some of the models we have proposed, for example, the stochastic models with absorbing states. In other models, however, this effect is a consequence of certain assumptions, for example, that players adjust their propensities of response as a result of experience. Even though the specific assumptions of the parameter-adjusting model (namely that the "system" passes through contiguous equilibria as in a reversible thermodynamic process) are not plausible, it is not unlikely that the effect is also a consequence of more general assumptions.