Chapter 6
General Remarks

IN THIS PART, we shall be constructing some mathematical models of the processes we have been describing. The task will remain unfinished, since the aim of a mathematical model ought to be to raise more questions than it answers. The value of such models is not confined to their explanatory power (of which some models have much and some little) but extends also to the theoretical leverage which these models provide. The nature of theoretical leverage will, we hope, become clear in a few illustrative examples.

One of the earliest numerical descriptions of a sequence of events is given in Genesis, where the vital statistics of the descendants of Adam are listed.

". . . . And all the days of Seth were nine hundred and twelve years; and he died.

"And Enoch lived ninety years and begat Kenan. . . . And all the days of Enoch were nine hundred and five years; and he died.

"And Kenan lived seventy years and begat Mahalel. . . ."

One can easily represent this account as a graph with the generations on the horizontal axis and longevity of the patriarchs on the vertical. However, one would not call the resulting graph a mathematical model, because it lacks theoretical leverage. No regularity is discernible in the numbers of years lived by the lineal descendants of Adam. At first it seems as if both longevity and the age of begetting the first son are diminishing, but this hypothesis is refuted almost immediately by Jared and again by Methuselah.

To be sure, the number of generations being finite, we could contrive to find a formula for passing a curve through all the ages. But such a curve would have as many free parameters as there are points to go through.[17] Again there would be no theoretical leverage, that is, no additional questions to be asked, except whether the formula continues to predict, which it probably does not.

At the other end of the scale are the classical physical laws, for example, the law of universal gravitation, the conservation laws, etc. Such laws are examples of models with very high explanatory power. The expression of these laws is terse; the consequences are extremely far-reaching.

A progression from purely descriptive models to models with high explanatory power often proceeds by reducing the freedom of the parameters used in the model. A good example would be such a progression in the development of the law of falling bodies, as it might have happened if the history of science confirmed more strictly our expectations in retrospect.

As is well known, Galileo established the law of motion for bodies undergoing constant acceleration, for example sliding down frictionless inclined planes. The distance traveled, s, is related to the time elapsed, t, by the formula

$$s = at^2 \qquad (31)$$

where a is constant. The constant is in this context a free parameter. Its value can be determined experimentally, and it has to be adjusted for every new inclined plane. The theoretical leverage of the model at this stage resides in this free parameter. For it soon becomes apparent that the magnitude of this parameter depends on the inclination of the plane. In fact, we are led to another law, namely

$$a = \tfrac{1}{2}g \sin \alpha \qquad (32)$$

where α is the angle between the inclined plane and the plumb line. Now g appears as a free parameter. The model still has some theoretical leverage, because investigations can proceed to determine relations between g and other variables. We now know that g depends on the distance from the center of the earth. The law of this dependence yields still another free parameter, namely K, the universal gravitational constant, which in turn can be investigated in the light of the current cosmological theories.

This idealized picture of progressive clarification suggests how one might proceed with the building of a theory, so that its succeeding stages are reached in the process of development. The method suggests also a scale of "strength" for models: the fewer free parameters a model has and the greater the range of situations to which it applies, the stronger it is.

Looking back now on the model proposed in Chapter 1 which purports to predict the relative frequencies of cooperative responses from the payoff matrices, we see that it is a rather weak one. It contains two free parameters and predicts only the rank order of games. However, the model is not totally devoid of theoretical leverage. It could be put to a test in any number of additional games, specifically constructed to test the model further. In particular, if the predicted positions of the games on the cooperative scale were checked out in the range examined, we could inquire about the limits of the range where the model remains valid, etc. Still, even in the most favorable circumstances, i.e., even if the predictions checked out in a large range, the model would still remain an extremely limited one.

We have seen that besides certain regularities in the relations between the C frequencies and the payoffs, there are also certain discernible time trends and certain correlations between members of pairs. Our game-index model implies nothing about these regularities. Clearly, if these are to be included in a theory, we need a model of the *process*, i.e., a dynamic model.

It stands to reason that interactions between the players exert important effects on the process, and it is these interactions which ought to be incorporated into a mathematical model. The derived consequences of the model ought to be statements about our variables as functions of time (i.e., of the number of plays) and about the way the variables are correlated.

Deterministic and Stochastic Models

Two broad classes of dynamic models can be distinguished, the deterministic (classical) and the stochastic. Each has certain advantages and certain disadvantages. The classical models are convenient, because they admit of well-known and established methods of classical analysis. Typically these models are represented by systems of differential equations.[18] It is easy enough to see how equations of this sort could be postulated for the process of repeated Prisoner's Dilemma games: they can represent the relations between the rates of change of the variables and their momentary values. When Lewis F. Richardson (1960) developed his dynamic theory of arms races, he did just that, namely, he assumed that the existing level of armaments of one nation acts as a stimulant to the rate of growth of armaments of a rival nation. Adding to these effects also some inhibitory terms to reflect the economic burden of armaments, and some constant terms to represent the chronic "grievances" of the nations against each other, Richardson derived for the case of two rival blocs a pair of differential equations with the interesting property that for certain values of the parameters the resulting system was stable while for certain other values it was unstable. The former kind of system could conceivably attain a balanced state of armament levels. The latter kind could only go to extremes, that is, it could either escalate into a runaway arms race or go into reverse toward complete disarmament. We could apply a similar conceptualization to the process we are studying.

An important shortcoming of the classical dynamic model is that it does not take into account the probabilistic features typical of behavioral phenomena. These features represent not merely compromises with our ignorance, as is always the case when determinate causes of events are not known. The probabilistic description of behavioral events is inherent in the very method of quantification of these events.

There is a fundamental distinction between mathematicized physical science and mathematicized behavioral science. In the former, the quantities appearing in equations refer typically to results of physical measurements, e.g., masses, velocities, concentrations, temperatures, etc. Also the early extension of physical science methods to psychology depended heavily on the isolation of appropriate physically measurable quantities of behavior. Such were, for example, reaction times, intensities of response reflected in physiological variables, etc. The gap between these measureable quantities and the usual descriptions of behavior is enormous, but the discrepancy does not necessarily have to do with whether behavioral events are described vaguely or precisely. There may be nothing vague about the statements "The rat passed a lever" or "Smith bought a car" or "player 1 played *C*" in the sense that independent observers will have no difficulty in agreeing on whether the event did or did not take place. Yet it is next to impossible to specify such events in terms of physically measured variables or to describe them minutely in terms of specific physical responses.

The objectivity of behavioral data depends on consensus among independent observers about what took place, not on the possibility of reducing the data to physically measurable variables. Once the objective nature of a unit event (e.g., a choice among specified alternatives) is established, a quantification of such events immediately suggests itself, namely in terms of their relative frequencies. Such a quantification lends itself admirably to mathematical treatment, first because frequencies can be objectively measured and second because a relative frequency is a pure number, independent of units. R. D. Luce (1959) has shown how the latitude of our choice of mathematical models depends vitally on the sort of scale used in the measurements. The weaker the scale, the more restricted is the class of models at our disposal. Relative frequencies are measured on an absolute scale, that is, a scale of pure numbers. This scale offers the widest possible range of possibly appropriate models.[19]

Now the relative frequency of an event can be operationally defined only with respect to a population of events. It cannot be defined with respect to a single event. A conceptual extension of the relative frequency notion to apply to a single event is one of the definitions of probability. The probability of a single event is thus a theoretical construct, never an observable. To be translated into observables, probabilities must first be transformed into frequencies.

A stochastic model of a process is concerned with the relations among various probabilities, particularly with the changes which such probabilities suffer as results of interactions of events.

Consider, for example, a sequence of events, such as a protocol of an individual's performance in Prisoner's Dilemma. Such a protocol will be a sequence of *C*'s and *D*'s. A stochastic model of this process would consist of some postulates about how the probability of a *C* response depends on what responses have been given before the play in question, including, generally, the responses of the other player. Consequently, the probability of a *C* response is treated in such a model as a variable, not a constant. It follows that this probability cannot be estimated from the relative frequency of *C* responses over a stretch of time, as could be done if this probability were constant. Such a probability could conceivably be estimated only if we had several "realizations" of this process, for example, if the game were played by several "identical" pairs of players (in the sense of obeying the "laws" of the same model). But even in this case, the estimation of the probability of a response would not be simple. For example, one could not simply take the relative frequency of *C* on the t-th response in a population of players as an estimate of the probability of the *C* response associated with "the" player. Even though the players of our hypothetical population may be "identical," they may have had different *histories* prior to the t-th response in the several realizations of the process, because the responses themselves were probabilistically determined, and consequently the probabilities of a response on a given play may all be different in the different "representations" of the player.

The matter is relatively simple if the probability of a response depends not on previous probabilities but on previous *events*. Suppose, for example, that the probability of a *C* depends only on what actually happened on the previous play, namely *CC*, *CD*, *DC*, or *DD*. If we have a large population of identical players, we can on a given play subdivide them into four subpopulations, namely those following a *CC* response, those following a *CD* response, etc. Each of these four subpopulations now has an identical "history" (since the history stretches no farther back than one previous play) and the probability of the *C* response can be estimated from its relative frequency in each subpopulation.

Clearly, the farther back the dependence on previous responses goes, the more such subpopulations have to be examined. Specifically their number grows exponentially with the length of the relevant "history." Very soon even enormously large populations of "identical" subjects become insufficient, because they have to be broken up into more subpopulations than there are individuals.

We have pointed out the difficulties involved in putting a stochastic model to a test—the difficulty of translating probabilities into frequencies, when the probabilities themselves depend on the realization of the stochastic process.

In short, the main difficulty in the use of stochastic models of behavior is in the fact that probabilities (which are the variables in such models) are elusive quantities. One never "measures" probabilities; one only estimates them. Typically the problems of estimating probabilities and the parameters governing the changes of probabilities become exceedingly involved in stochastic models where interactions occur.

Interactions are the very essence of the Prisoner's Dilemma played repeatedly; consequently the stochastic models of this process become exceedingly involved. We shall propose some models of this sort, but we shall not put them to tests by the usual method of estimating parameters and comparing predicted with observed results. Instead we shall content ourselves with trying to reproduce some of the gross characteristics of our data by simulation, i.e., postulating players governed by the models to whom some arbitrarily selected values of the parameters are ascribed.

We shall also propose some classical (deterministic) models, which have lately passed out of fashion in mathematical psychology. We shall make no claim for the adequacy of such models. Our purpose in introducing them will be merely to see whether we can derive some gross features of our data and, if so, whether we can draw some suggestive hypotheses about the psychology of the process from the way the theoretically derived results depend on the parameters.