Prisoner's Dilemma
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Chapter 7
Markov Chain Models

THE FIRST MODELS we shall examine will be Markov chains. In these models it is assumed that the system under consideration can be at any given time in any of a finite number of states. Let this number be n. When a system is in a given state, say s_i, it can pass to another state, say s_j, with a certain transition probability α_{ij}. Consequently, there are n^2 quantities α_{ij} $(i, j = 1, 2 \ldots n)$ which can be put into a square matrix of order n. The transition probability α_{ij} is the entry in the i-th row and the j-th column of this matrix. Some of the α_{ij} may be zero, which means that the system never passes from the particular state s_i to the particular state s_j. Some may be equal to 1, which means that whenever the system is in some particular state s_i it always passes from it to the particular state s_j, etc.

If some α_{ii} equals 1, then the system, once it finds itself in s_i will stay in it forever. Such a state is called an *absorbing state*. In a later chapter we shall be concerned with Markov chains in which there are absorbing states, but for the time being we shall be concerned only with chains in which it is possible to pass from any state to any other state at least via intermediate states. Clearly, such chains have no absorbing states.

More specifically, we shall be first concerned with ergodic chains. Roughly, an ergodic chain is one in which ultimately the system will "visit" each of the states with a certain constant frequency (but not strictly periodically). This frequency is independent of the initial probability of finding the system in any state.

Time, in the context of Markov chains, is quantized. That is, the moments of time are represented by the transition steps from state to state, and so the time variables take on discrete values: $t = 0,1,2 \ldots$.

Let $p_j(t)$ represent the probability that at time t the system is in state s_j. Then clearly

$$p_j(t+1) = \sum_{i=1}^{n} \alpha_{ij}\, p_i(t) \quad (j = 1,2 \ldots n). \qquad (33)$$

Equation (33) describes the time course of the probability distribution $p_j(t)$. If the initial probability distribution $p_j(0)$ is known, then the fate of the system is determined with respect to the probabilities that the system is in one of the possible states at any future time but, of course, not with respect to the actual sequence of states to be traversed. The actual time course will be a "realization" of any one of the possible sequences of which there are a great many.

If the chain is ergodic, the distribution $p_j(t)$ tends to a limiting distribution $p_j(\infty)$. This means that after a sufficiently long time, the system will be "visiting" each of the states s_j with relative frequency $p_j(\infty)$. These limiting frequencies can be obtained by setting $p_j(t+1) = p_j(t)$ in Equation (33) and solving for the p_j $(j = 1 \ldots n)$.[20] For the time being, we shall be concerned only with these equilibrium distributions, the so-called *steady states* of Markov chains.

Let the probability of a cooperative response of a player depend only on what happened on the previous play. The dependence can be (1) on what the player in question did, (2) on what the other player did, or (3) on what both of them did. If the dependence is only on what the player in question did, clearly our conditional probabilities η and ζ are the relevant parameters (cf. p. 67). If the dependence is only on what the other player did, then the relevant parameters are ξ and ω. If the dependence is on what both did, we must use x, y, z, and w as our parameters (cf. p. 71).

Of these three cases, the first is not interesting. If the response probabilities of each player are determined only by what he himself has been doing, there is no interaction between the two. The essence of Prisoner's Dilemma is in the interactions between the pair members. To be sure, the noninteractive model cannot be discarded on a priori grounds alone. However, since we have already seen how strong positive interactions are reflected in the data (cf. Chapter 3), we can dismiss the models governed by η and ζ alone. We already know that they are inadequate to describe the results of our experiments.

We turn to a model which supposes that the process is governed by ξ and ω. We wish to examine the simplest possible types first in order to see what interesting features of the process, if any, are reflected already in the simplest models.

The most drastic simplification of a (ξ, ω) model would result if one left only one of these parameters free and assigned an extreme value, zero or one, to the other. However, some of these models are immediately revealed as trivial, as we shall now show.

Let ξ be arbitrary and $\omega = 0$. This means that in response to the other's defection, a player always defects, and in response to the other's cooperation he responds cooperatively with probability ξ. Since $0 < \xi < 1$, a double defection is bound to occur sometime. Further, because $\omega = 0$, this double defection will be immediately fixated and will persist thereafter.

A similar argument reveals the triviality of the model where $0 < \omega < 1$ and $\xi = 1$, since in that case a *CC* response will be certainly fixated.

On the other hand, the two models represented by

$0 < \xi < 1; \omega = 1$ and by $0 < \omega < 1; \xi = 0$ lead to non-trivial steady states which depend on the respective variable parameters. Let us consider each of these in turn.

Case: $0 < \xi < 1; \quad \omega = 1.$

In this situation, each player responds cooperatively whenever the other defects and with respective probabilities ξ_1 and ξ_2 whenever the other cooperates. This assumption may seem unrealistic, but can be rationalized. When the other player defects, the player in question tries to induce him to cooperate, hence he himself cooperates. When the other cooperates, the player in question sometimes cooperates (with probability ξ_i) but is sometimes tempted to defect (with probability $1 - \xi_i$). In any case, the realism of this assumption will not be of concern to us in these preliminary theoretical explorations.

The Markov equations are now the following:

$$(CC)' = CC\xi_1\xi_2 + CD\xi_2 + DC\xi_1 + DD, \tag{34}$$

$$(CD)' = CC\xi_1(1 - \xi_2) + CD(1 - \xi_2), \tag{35}$$

$$(DC)' = CC(1 - \xi_1)\xi_2 + DC(1 - \xi_1), \tag{36}$$

$$(DD)' = CC(1 - \xi_1)(1 - \xi_2), \tag{37}$$

where the primed quantities represent the probabilities of the corresponding states on the play following the play in question.

To see this, denote (CC) by s_1, (CD) by s_2, (DC) by s_3, and (DD) by s_4. Observe that the transition probability α_{11} is given in the present model by $\xi_1\xi_2$, since when the system is in state s_1 it passes to the same state if and only if both players respond cooperatively, and the conditional probability of this event is $\xi_1\xi_2$ under the assumption that in the absence of communication the responses of the two subjects in any particular play must be independent of each other. Similarly $\alpha_{21} = \xi_2$, since when the system is in state *CD* (s_2) we have assumed that player 1 will always cooperate (in response to the other's defection, since $\omega = 1$) while the second player will cooperate with probability ξ_2, etc.

The steady state solution of the four categories of responses is given by

$$CC = \frac{\xi_1\xi_2}{(\xi_1 + \xi_2)^2 + \xi_1\xi_2(\xi_1\xi_2 - 2\xi_1 - 2\xi_2)}; \quad (38)$$

$$CD = \frac{\xi_1^2(1 - \xi_2)}{(\xi_1 + \xi_2)^2 + \xi_1\xi_2(\xi_1\xi_2 - 2\xi_1 - 2\xi_2)}; \quad (39)$$

$$DC = \frac{\xi_2^2(1 - \xi_1)}{(\xi_1 + \xi_2)^2 + \xi_1\xi_2(\xi_1\xi_2 - 2\xi_1 - 2\xi_2)}; \quad (40)$$

$$DD = \frac{\xi_1\xi_2(1 - \xi_1)(1 - \xi_2)}{(\xi_1 + \xi_2)^2 + \xi_1\xi_2(\xi_1\xi_2 - 2\xi_1 - 2\xi_2)}. \quad (41)$$

It is easily verified that *CC* tends to 1 and all the other states to zero as ξ_1 and ξ_2 tend to one. When ξ_1 and ξ_2 vanish, the expression becomes indeterminate, but limits can be evaluated if the way in which each ξ_1 approaches zero is specified.[21] In particular, if $\xi_1 = \xi_2$ as both approach zero, the limiting distribution is $CC = CD = DC = DD = \frac{1}{4}$, as can be seen intuitively, for in that case, if the players happen to start in either *CD* or *DC*, they will remain in that state, and if they happen to start with either *CC* or *DD*, they will oscillate between these two states.

This model has one feature which immediately removes it from consideration. In the steady state, the correlation coefficient ρ_0 (cf. p. 67) must vanish, regardless of the values of ξ_1 and ξ_2. This is so because the numerator of the expression denoting ρ_0 [Equation (10) of Chapter 3] vanishes when the expressions (38-41) are substituted for the four states.

Case: $\xi_1 = \xi_2 = 0; \quad 0 < \omega_1 < 1; \quad 0 < \omega_2 < 1.$

In this case, the Markov equations for the steady state become

$$CC = DD\omega_1\omega_2; \tag{42}$$

$$CD = CD\omega_1 + DD\omega_1(1 - \omega_2); \tag{43}$$

$$DC = DC\omega_2 + DD\omega_2(1 - \omega_1); \tag{44}$$

$$DD = CC + CD(1 - \omega_1) + DC(1 - \omega_2) + DD(1 - \omega_1)(1 - \omega_2). \tag{45}$$

Proceeding exactly the same way, we get the steady-state distribution:

$$DD = \frac{(1 - \omega_1)(1 - \omega_2)}{(1 - \omega_1)(1 - \omega_2)(1 + \omega_1\omega_2) + \omega_1(1 - \omega_2)^2 + \omega_2(1 - \omega_1)^2}; \tag{46}$$

$$DC = \frac{\omega_2(1 - \omega_1)^2}{(1 - \omega_1)(1 - \omega_2)(1 + \omega_1\omega_2) + \omega_1(1 - \omega_2)^2 + \omega_2(1 - \omega_1)^2}; \tag{47}$$

$$CD = \frac{\omega_1(1 - \omega_2)^2}{(1 - \omega_1)(1 - \omega_2)(1 + \omega_1\omega_2) + \omega_1(1 - \omega_2)^2 + \omega_2(1 - \omega_1)^2}; \tag{48}$$

$$CC = \frac{\omega_1\omega_2(1 - \omega_1)(1 - \omega_2)}{(1 - \omega_1)(1 - \omega_2)(1 + \omega_1\omega_2) + \omega_1(1 - \omega_2)^2 + \omega_2(1 - \omega_1)^2}. \tag{49}$$

Here *DD* tends to one as ω_1 and ω_2 tend to zero. As ω_1 and ω_2 tend to one (while $\omega_1 = \omega_2$), the system again tends toward equal distributions of states. Again we see that $\rho_0 = 0$ for any values of ω_1 and ω_2. Hence this model also fails to account for the observed predominantly positive ρ_0.

We could go on to a general model of this sort with arbitrarily chosen values of the eight parameters (subject, of course, to the constraints noted on p. 34). However, the expressions become unwieldly in the general case, and we shall not develop these types of models further. Instead, we turn our attention to models based on the state-conditioned propensities, x, y, z, and w.

Case:

$$0 < x_1 < 1; \quad 0 < x_2 < 1; \quad y = z = 0; \quad w_1 = w_2 = 1.$$

This will be recognized as the case of the two tempted simpletons (cf. p. 85). The simpletons always change their responses when the payoff is negative and stay with the same response when the payoff is positive except that following a *CC* response, each may defect with probability $1 - x_i$ ($i = 1, 2$). Consequently, the *CD* and the *DC* responses are always followed by *DD*, which, in turn is always followed by *CC*. However, the *CC* response can be followed by any of the other three.

The steady-state Markov equations for this case are (cf. Note 14)

$$CC = CCx_1x_2 + DD; \tag{50}$$

$$CD = CCx_1(1 - x_2); \tag{51}$$

$$DC = CCx_2(1 - x_1); \tag{52}$$

$$DD = CC(1 - x_1)(1 - x_2) + CD + DC. \tag{53}$$

The steady-state distribution is (cf. p. 75)

$$CC = \frac{1}{2 + x_1 + x_2 - 3x_1x_2}; \tag{54}$$

$$CD = \frac{x_1(1 - x_2)}{2 + x_1 + x_2 - 3x_1x_2}; \tag{55}$$

$$DC = \frac{x_2(1 - x_1)}{2 + x_1 + x_2 - 3x_1x_2}; \tag{56}$$

$$DD = \frac{1 - x_1x_2}{2 + x_1 + x_2 - 3x_1x_2}. \tag{57}$$

Let us now compute the numerator of the expression for ρ_0. This is

$$\begin{aligned}(CC)(DD) - (CD)(DC) &= (1 - x_1x_2) - (x_1x_2 - x_1^2x_2 - x_1x_2^2 + x_1^2x_2^2) \\ &= 1 - 2x_1x_2 + x_1^2x_2 + x_1x_2^2 - x_1^2x_2^2. \end{aligned} \tag{58}$$

Consider now the polynomial on the right side of (58) which we shall call $F(x_1,x_2)$. We shall show that the value of this polynomial for all values of x_1 and x_2 in the interval $(0 < x < 1)$ is positive and consequently that ρ_0 is positive for all values of x_1 and x_2 in that interval.

First observe that the partial derivatives of $F(x_1,x_2)$ with respect to x_1 and x_2 are both negative in the interval $(0 < x < 1)$. For

$$\frac{\partial F}{\partial x_1} = -2x_2 + 2x_1x_2 + x_2^2 - 2x_1x_2^2$$
$$= x_2[(x_1 - 1)(1 - x_2) + x_1 - 1 - x_1x_2] < 0; \quad (59)$$

$$\frac{\partial F}{\partial x_2} = -2x_1 + x_1^2 + 2x_1x_2 - 2x_1^2x_2$$
$$= x_1[(x_2 - 1)(1 - x_1) + x_2 - 1 - x_1x_2] < 0, \quad (60)$$

if $0 < x_1 < 1; 0 < x_2 < 1$. This means that $F(x_1,x_2)$ increases as either x_1 or x_2 decreases when x_1 and x_2 are in the interval $(0,1)$. But $F(1,1) = 0$,[22] $F(0,0) = 1$. Therefore $F(x_1,x_2) > 0$ in the interval $(0 < x < 1)$, and so is ρ_0.

We have now shown how a positive correlation of C responses (and therefore of D responses) is a consequence of the simplest interactive model, based on the conditional probabilities x, y, z, and w (namely the very special case where $y = z = 0$; $w = 1$). We therefore select this model for further treatment.

The Four-State Markov Model, General Case

We now assume that x_i, y_i, z_i, and w_i $(i = 1, 2)$ can assume arbitrary values in the interval $(0,1)$. The complete set of Markov equations is now given by

$$(CC)' = CCx_1x_2 + CDy_1z_2 + DCy_2z_1 + DDw_1w_2, \quad (61)$$
$$(CD)' = CCx_1\tilde{x}_2 + CDy_1\tilde{z}_2 + DC\tilde{y}_2z_1 + DDw_1\tilde{w}_2, \quad (62)$$
$$(DC)' = CC\tilde{x}_1x_2 + CD\tilde{y}_1z_2 + DCy_2\tilde{z}_1 + DD\tilde{w}_1w_2, \quad (63)$$
$$(DD)' = CC\tilde{x}_1\tilde{x}_2 + CD\tilde{y}_1\tilde{z}_2 + DC\tilde{y}_2\tilde{z}_1 + DD\tilde{w}_1\tilde{w}_2, \quad (64)$$

where we have written $\tilde{x}$ for $1 - x$, $\tilde{y}$ for $1 - y$, etc.

The steady-state equations are derived in the same way as previously, namely by setting $(CC)' = CC$, $(CD)' = CD$, etc., and by solving the resulting system for the four states. The solution is extremely unwieldly and is omitted here. The special case where $x_1 = x_2$, $y_1 = y_2$, etc., is shown below.

$$(CC) = \frac{w^2(1 - y\tilde{z} - \tilde{y}z + 2w\tilde{w}) - 2w\tilde{w}(w^2 - yz)}{(1 - y\tilde{z} - \tilde{y}z + 2w\tilde{w})(1 - x^2 + w^2) - 2(w^2 - yz)(w\tilde{w} - x\tilde{x})}, \quad (65)$$

$$(CD) = \frac{w\tilde{w}(1 - x^2 + w^2) - w^2(w\tilde{w} - xx)}{(1 - y\tilde{z} - \tilde{y}z + 2w\tilde{w})(1 - x^2 + w^2) - 2(w^2 - yz)(w\tilde{w} - x\tilde{x})}, \quad (66)$$

$$(DC) = (CD), \quad (67)$$

$$(DD) = 1 - (CC) - 2(CD). \quad (68)$$

Equations (61)-(64) constitute the four-state Markov chain model. To put such a model to a test one needs to estimate the parameters x_i, y_i, z_i, and w_i, as well as the initial probability distribution of the four states. If such estimates can be obtained, statistical predictions can be made to be compared with the statistics obtained from the data. As we have already pointed out, the task of estimating the parameters is by no means easy. Besides, a single set of estimates will enable us to test the model only in a single situation. As a consequence, even if the test corroborates the model we shall still be in the dark concerning the generality of its applicability. Therefore, rather than investigate a single model in detail, we shall develop a variety of models. Our aim is to indicate many different approaches and to postpone the question which of them, if any, is fruitful. In what follows, therefore, we shall describe other mathematical models of Prisoner's Dilemma which seem reasonable to us. When we finally undertake the problem of testing the models it will be only with respect to certain general features of the data with a view of demonstrating the sort of questions which naturally arise in the light of the mathematical models. In this book no definitive argument will be made in favor of any of the models, although evidence for and against some of the models we have constructed will be offered.

Markov Model with Absorbing States

An absorbing state is one from which there is no exit. Once a system enters this state, it remains in it thereafter. Such states are well known in nature, for example the end states of irreversible chemical reactions, death, considered as a state of an organism, etc. In our contexts, absorbing states would be introduced if we supposed that at times one or both players make a decision to play exclusively cooperatively or exclusively uncooperatively, no matter what happens. However, as we shall see, absorbing states can be introduced also under a weaker assumption.

A successful application of an absorbing state model was made by Bernard P. Cohen (1963) in another experimental context. Cohen's experiments were variants of Asch's experiments on conformity under social pressure.

The essential feature of the situation examined by Asch and Cohen is that a single subject is required to make judgments (about comparative lengths of line segments) following the expressed judgments of several pseudo-subjects, who are actually confederates of the experimenter. These pseudo-subjects make deliberately false judgments about the relative lengths of lines, which ordinarily would be easily judged correctly. This presumed social pressure frequently induces the bonafide subject to make incorrect judgments also.

Cohen's model of the process is a Markov chain with four states, two of which are absorbing states. The states are:

s_1: if the subject is in this state, he will give the correct response on the next trial and correct responses thereafter.

s_2: if the subject is in this state, he will give the correct response on the next trial but may give wrong responses on subsequent trials.

s_3: if the subject is in this state, he will give the wrong response on the next trial but may give correct responses on subsequent trials.

s_4: if the subject is in this state, he will give the wrong response on the next trial and on every trial thereafter.

The confederates are instructed always to give the wrong response following the first two trials (on which correct responses are given "to establish confidence" that the lengths are indeed being compared).

Consequently, a decision on the part of the subject to give wrong responses is behaviorally equivalent to a decision to conform to the judgment of the group. Similarly, a decision to give correct responses is equivalent to a decision to ignore the judgment of the group. If a subject makes either of these commitments, he passes into one of the two absorbing states (s_1 or s_4).

Now a Markov chain with two absorbing states will eventually pass into the one or into the other. If it is possible to pass into either of the two absorbing states from an arbitrary nonabsorbing state, there is no way of knowing with certainty in which of the absorbing states the system will end up. Such a process is non-ergodic. This means roughly that a given realized history of the system will not exhibit the same relative frequencies of states independent of initial conditions (as is the case with ergodic processes). It follows that the process described by Cohen's model is not an ergodic process. Any realization of it (a protocol) is bound to end up in one or in the other of the absorbing states. However, the probability of ending up in one or the other can be determined from the initial condition (which Cohen assumes with justification to be s_2) and from the transition probabilities. The latter must be estimated from the data.

The circumstance which led Cohen to postulate a model with absorbing states was the fact that several of the subjects did end up in uninterrupted runs of correct or incorrect responses. Thus Cohen observed essentially the same sort of lock-in effect which we observed in repeated Prisoner's Dilemma. Drawing the statistical inferences from the resulting stochastic model, Cohen got good agreements with several of the important statistics of the process (though not all).

Let us now construct a Markov model with absorbing states for Prisoner's Dilemma. The relevant "decisions" immediately suggest themselves, namely a decision on the part of a subject henceforth to cooperate or a decision henceforth not to cooperate. Theoretically, these decisions could be made unconditionally, namely to cooperate no matter what the other does or not to cooperate regardless of what the other does. Our situation, however, speaks against such a hypothesis. We must remember that we have here two minds, not one. Therefore each of the subjects can make his decision independently of the other. If we supposed that one subject makes a decision to cooperate no matter what the other does, and the other makes an opposite decision, we would observe sessions ending in runs composed exclusively of *CD* or *DC*. Now long runs of unilateral cooperation are sometimes observed, but they are rather rare, and the ending of a session in such a run almost never occurs. We can, however, easily exclude the possibility of the state in which one of the players has decided irrevocably to cooperate while the other has decided irrevocably to defect. We need only assume that passage to the absorbing states, i.e., the irrevocable decision, is possible only when the system is in the *CC* state (in which state one or both of the players may decide to cooperate from then on) or in the *DD* state (in which state one or both may decide not to cooperate from then on). It follows that opposite decisions cannot be made. Obviously they cannot be made simultaneously. Nor can they be made on different occasions. For suppose player 1 has already made the decision to cooperate. Then the *DD* state will never occur, and consequently, by our hypothesis player 2 cannot make the irrevocable decision to defect. Similarly, if one player decides always to defect, the *CC* state will never occur; consequently, the other player cannot make the opposite decision.

We shall now construct a matrix of transition probabilities for one of the players. The entries in this matrix, however, will be not single transition probabilities but pairs of such, because what the subject is prone to do depends not only on what he himself has just done but also on what the other player did on the previous play. Let the states of the single player be as follows:

Γ: if the player is in this state, he will henceforth play only *C*.

C: if the player is in this state, he will play *C* but may play *D* on succeeding plays.

D: if the player is in this state, he will play *D* but may play *C* on succeeding states.

Δ: if the player is in this state, he will henceforth play only *D*.

Matrix 14 shows the transition probabilities. The first of each pair denotes the transition probability in case the other player has played *C*; the second, in case he has played D. Note that the transition probabilities cannot depend on the *state* of the other player. His state is known only to himself. His partner can observe only his choice on the preceding play.

	Γ	C	D	Δ
Γ	1,1	0,0	0,0	0,0
C	γ,0	x,y	$1-x-\gamma,\ 1-y$	0,0
D	0,0	z,w	$1-z,\ 1-w-\delta$	0,δ
Δ	0,0	0,0	0,0	1,1

Matrix 14.

	$\Gamma\Gamma$	ΓC	ΓD	$\Gamma\Delta$	$C\Gamma$	CC	CD	$C\Delta$	$D\Gamma$	DC	DD	$D\Delta$	$\Delta\Gamma$	ΔC	ΔD	$\Delta\Delta$
$\Gamma\Gamma$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ΓC	γ	x	$\tilde{x}$	0	0	0	0	0	0	0	0	0	0	0	0	0
ΓD	0	z	$\tilde{z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Gamma\Delta$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$C\Gamma$	γ	0	0	0	x	0	0	0	$\tilde{x}$	0	0	0	0	0	0	0
CC	γ^2	γx	$\gamma\tilde{x}$	0	γx	x^2	$x\tilde{x}$	0	$\tilde{x}\gamma$	$\tilde{x}x$	$\tilde{x}^2$	0	0	0	0	0
CD	0	0	0	0	0	yz	$y\tilde{z}$	0	0	$\tilde{y}z$	$\tilde{y}\tilde{z}$	0	0	0	0	0
$C\Delta$	0	0	0	0	0	0	0	y	0	0	0	$\tilde{y}$	0	0	0	0
$D\Gamma$	0	0	0	0	z	0	0	0	$\tilde{z}$	0	0	0	0	0	0	0
DC	0	0	0	0	0	zy	$z\tilde{y}$	0	0	$\tilde{z}y$	$\tilde{z}\tilde{y}$	0	0	0	0	0
DD	0	0	0	0	0	w^2	$w\tilde{w}$	$w\delta$	0	$\tilde{w}w$	$\tilde{w}^2$	$\tilde{w}\delta$	0	δw	$\delta\tilde{w}$	δ^2
$D\Delta$	0	0	0	0	0	0	0	w	0	0	0	$\tilde{w}$	0	0	0	δ
$\Delta\Gamma$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ΔC	0	0	0	0	0	0	0	0	0	0	0	0	0	y	$\tilde{y}$	0
ΔD	0	0	0	0	0	0	0	0	0	0	0	0	0	w	$\tilde{w}$	δ
$\Delta\Delta$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Matrix 15.

Assuming identical players, we can now construct the matrix of transition probabilities for the pair. This is shown in Matrix 15.

Matrix 15 constitutes the Markov chain model with absorbing states. As we see, it turns out to be a fourteen-state Markov chain if transitions to the absorbing states can be made only from *CC* and *DD*. (Otherwise there are sixteen states.)

Here $\tilde{y}$ and $\tilde{z}$ denote $1 - y$ and $1 - z$ as usual, but $\tilde{x}$ and $\tilde{w}$ denote $1 - x - \gamma$ and $1 - w - \delta$ respectively. Tests of this model by simulation methods will be discussed in Chapter 12.