A General Social Dilemma: Profitable Exchange and Intransitive Group Preferences

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During the last few years two seemingly unrelated developments have aroused attention among many economists, political scientists, and sociologists. The first is a general proof that the possibility of a profitable exchange of a majority of votes in a system of majority voting implies cyclical or at least intransitive group preferences and vice versa, if separable individual preferences are present (Bernholz 1973, 1974*a;* Oppenheimer 1972; Miller 1977; Schwartz 1977). The second is the proof first given by Sen that minimal liberalism and Pareto optimality may lead to a contradiction (Sen 1970). This implies, however, that with independent decisions by some individuals an outcome results which is not Pareto-optimal. As a consequence, these individuals could agree on profitable exchange to reach a Pareto-optimal outcome. Thus again, if a profitable agreement among some members of a group is possible, cyclical group preferences are present.

This result is preserved if a definition of minimal liberalism, which seems more adequate than Sen's, is accepted. Whereas in the latter definition at least two different individuals have the right to decide among two different pairs of outcomes, the former definition gives the right to individuals to decide among the alternatives of different issues or characteristics of outcomes (Bernholz 1974*b,* 1975; Gibbard 1974; Breyer 1978).

It has been shown that the logrolling theorem stated above is not only true for simple majority voting but also for all kinds of qualified majority voting up to near-unanimity (Bernholz 1974*a;* Oppenheimer 1972). The relation between the logrolling theorem and the paradox of liberalism, moreover, has soon been recognized (Bernholz 1976). The most general proof until now of the logrolling theorem including liberalism and all other kinds of social organization or allocations of rights to make decisions in a society, which will be discussed below, has been given by Schwartz (1977) for the case of separable individual preferences. But it is not clear from Schwartz's paper what kinds of assignments of rights are included and what the logrolling theorem means for the corresponding social organizations.

Let us sum up. It has been suggested by the developments sketched above that whenever a situation (outcome) results from the independent decisions of the subgroups who have the right to decide different issues which can be overturned by and to the benefit of a winning coalition of these subgroups,

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then intransitive social preferences are present, if individual preferences are separable (compare sec. 2c). The very general meaning of this result as a social dilemma has, however, not been grasped until now and it is still confined to the assumption of separable individual preferences. It is the task of the present discussion to remove the latter assumption and to clarify the meaning of the general theorem.

Next, there is a close relationship between Arrow's General Impossibility Theorem and our results, as will be shown in section 3. For our assumptions fulfill Arrow's conditions. Thus the relations stated in this paper can be looked at as a substantive interpretation of Arrow's theorem.

Finally, it will be shown that the ''paradox'' proved by Sen for liberalism appears to be true for nearly all the decentralized social organizations considered.

1. Generalization of the Results Obtained for Majority Voting to a General Class of Decision Rules and Assignments of Rights

a. Definitions and Notation

In the following section, we consider a set V of $m \geqq 2$ individuals with weak, ordinal, complete and transitive preferences and a set M of $n \geqq 2$ issues M_i ($i = 1, 2, \ldots, n$). Each issue M_i comprises a finite number $m_i \geqq 2$ of alternatives. An outcome $x \in U$ is an n-dimensional vector with one and only one component (alternative) out of each issue. Consequently $U = M_1 \times M_2 \times \ldots \times M_n$. Note that there are $m_1 \cdot m_2 \cdot m_3 \cdot \ldots \cdot m_n$ outcomes.

For each issue M_i, there is a nonempty set $V_i \subseteq V$ of individuals, who have the right to decide according to some nonstochastic decision rule d_i for society among the alternatives of M_i. Note that only one decision rule d_i is used for issue M_i. Otherwise contradictory results might be reached for M_i.

Decision rule d_i ($i = 1, 2, \ldots, n$) can be any decision rule having the following properties: Assume that $x, y \in U$ are any two outcomes differing only in issue M_i ($i = 1, 2, \ldots, n$), and that we have any combination of individual preference orderings over them, namely $x R_j y$, $y R_j x$ or $x R_j y$ *and* $y R_j x$ for all $j \in V - V_i$. Then

a) there exists at least one $L_i \subseteq V_i$ ($L_i \neq \emptyset$) for any given combination of individual preferences over x and y for all $s_i \in V_i - L_i$ such that if for all $j_i \in L_i$ $x P_{ji} y$ is true, society prefers x to y, i.e. $x P y$;
b) if $L_i \subset V_i$ then a) is true for $L_i \cup \{g\}$, $g \in V_i - L_i$, if the same combination of individual preferences is given for all $s_i \in V_i - (L_i \cup \{g\})$ and if $x P_{ji} y$ for all $j_i \in L_i$ and $x P_g y$;
c) L_i and $L_i \cup \{g\}$ can bring about x as against y by using d_i, if only a decision is made concerning these outcomes.

Each L_i for all of whose members $x P_j y$ is true and which for a given decision rule d_i implies $x\,P\,y$, will be called a winning coalition C_i. Obviously there may be quite a number of winning coalitions C_i, C_i', C_i'', . . . for any pairs of alternatives.

It is obvious from the above that there may exist winning coalitions which are able to decide among z and v in their favour, $z, v \in U$, if these outcomes are different in exactly h well-determined issues, $1 \leqq h \leqq n$. We call such a coalition an h-issue coalition C^h. C_i, C_i', C_i'' are one-issue coalitions. C^h is composed out of h one-issue coalitions, e.g. $C^h = C_1 \cup C_2 \cup C_3 \cup \ldots \cup C_h$. There may be quite a number of such C^h coalitions, which may refer to the same issues and pairs of alternatives, but also to different pairs of alternatives and to different bundles of h issues. For our purpose we need, however, not make a notational difference between them. We will sometimes write ${}^rC^h$ ($r = 1, 2, \ldots$) to point out that different h-issue coalitions exist.

Let us stress that the above assumptions allow a wealth of different institutional arrangements of society. If d_i provides for majority rule among V_i for all i and if $V_i = V$, then we have a society in which all issues are decided by a majority of all voters. The other extreme case, "pure liberalism," would be present, if $V_i = \{j_i\}$, $j_i \in V$, and if each individual j_i would have the right to decide at least one issue (thus $m \leqq n$). The meaning of d_i would, of course, be quite obvious in this case. As an additional case, $V_1 = V_2 = \ldots = V_n = \{j\}$ would be a pure dictatorship.

But there can be, too, all possible combinations of allocations of issues to different subgroups V_i and of different decision rules. E.g., there may be issues $M_1, \ldots, M_m$ each of which has been allocated to a different individual of society according to the principle of liberalism; issues $M_{m+1}, \ldots, M_r$ may be decided by majority rule with $V_{m+1} = V_{m+2} \ldots = V_r = V$; M_{r+1} may be decided by a set of bureaucrats $V_{r+1} \subset V$ according to some decision rule d_{r+1}; M_{r+2} by a set of shareholders deciding $V_{r+2} \subset V$ according to a decision rule d_{r+2}, stating that all shareholders owning a majority of shares form winning coalitions C_{r+2}, C'_{r+2}, C''_{r+2}, . . . etc. Note, however, that we do not allow d_i to be a chance mechanism, since we have confined our analysis to nonstochastic decision rules.

We conclude by completing our definitions concerning social preference relations. Let us reçall first that "society prefers x to y", $x\,P\,y$, if there exists at least one winning coalition C for all of whose members $j \in C$ $x\,P_j\,y$ holds.

Next, let us assume that for the moment only 1-, 2-, . . . , h-issue coalitions are allowed. Then, if some x and y are different in $k > h$ issues no k-issue coalition can be formed. This means, of course, that there exists no subset of society $V^k \equiv V_1 \cup V_2 \cup \ldots \cup V_k \subseteq V$ having the right to decide among x and y.

As a consequence there can be no winning k-issue coalition C^k, too, whatever the preferences of the members of V. Society is not able or willing to form collective preferences or to make collective decisions between x and y. Thus we will say that no social preference relation exists between x and y, if no k-issue coalitions are allowed. The social preference relation is incomplete, since there is no social preference relation between all possible pairs of outcomes in U.

Finally, let some x and y differ only in at most h issues and let 1-, 2-, . . . , h-issue coalitions be allowed. Then $x\ R\ y$, "society prefers x to y or is indifferent between them," if there is no winning coalition whose members prefer y to x, i.e. if $y\,Px$ does not hold, with $x, y \in U$ and different in at most h issues. We may also say that there exists a blocking coalition preventing y. If neither xPy nor yPx are valid, i.e., if neither a winning coalition for x against y nor for y against x exists, again with $x, y \in U$ and different in at most h issues this will be expressed by $x\ I\ y$, "society is indifferent between x and y." This could, e.g., easily happen if x and y were different only in issue M_i and if d_i would prescribe a two-thirds majority of the members of V_i for a valid decision. Again, $x\ R\ y$ is written if either $x\ P\ y$ or $x\ I\ y$ is true. Thus either a winning coalition preferring x over y exists or no winning coalition preferring x over y or y over x is present. Obviously, if $x\ R\ y$ and $y\ R\ x$ are both true, then $x\ I\ y$.

b. Representation by a Digraph[1]

A finite directed graph (or finite digraph) consists of a finite number of points or vertices, part or all of which are connected by directed lines (arrows). If each pair of points is connected by at least one arrow one speaks of a complete digraph. In our case there are $m_1 \cdot m_2 \cdot \ldots \cdot m_n$ points, namely all possible outcomes. The directed lines represent the R-relations between outcomes. If, e.g., $x\ R\ y$, then an arrow leads from x to y (see fig. 1). If for $u, v \in U$, $u\ R\ v$ and $v\ R\ u$ is true u and v are connected by an arrow from u to v and by another one from v to u. Two arrows between two vertices in opposite directions thus symbolize the indifference relations, here $u\ I\ v$. From this, if $x, y \in U$ are only connected by one arrow leading from x to y then $x\ P\ y$ holds.

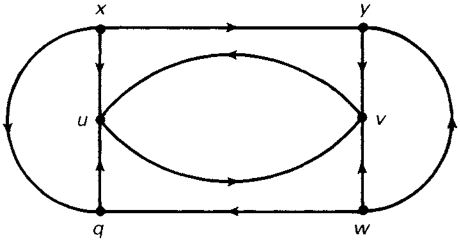

Fig. 1

Next we define what we call a digraph D_h $(1 \leqq h \leqq n)$. Assume that only $C^1=$, $C^2=$, . . . , C^h-issue coalitions are allowed or considered. Then, for $h < k \leqq n$ no arrows exist between outcomes (points) x and w which are different in k, i.e., in more than h issues. For there are no k-issue coalitions possible. Thus no R-relation and no directed line is present between x and w.

It follows from this definition that only digraph D_n is a complete digraph. All other digraphs D_h $(1 \leqq h < n)$ are incomplete digraphs, since not all pairs of outcomes are connected by arrows. Moreover, each digraph D_h is a subgraph of each digraph D_k for $h < k$, since a digraph D_h has the same $m_1 \cdot m_2 \cdot \ldots \cdot m_n$ outcomes (points) as a digraph D_k and only part of the arrows present in the latter. The digraph in figure 1 can be interpreted as a digraph D_1, if we define $x = (x_1, y_1)$, $u = (x_2, y_1)$, $q = (x_3, y_1)$, $y = (x_1, y_2)$, $v = (x_2, y_2)$ and $w = (x_3, y_2)$ as two-dimensional vectors. Since only one-issue coalitions are allowed, no directed lines between points which are different in two issues are present. Thus there are no arrows between, e.g., x and w and u and y. The digraph is incomplete. The complete digraph corresponding to the digraph D_1 of figure 1 is given by the digraph D_2 of figure 2, because $n = 2$ in this case. One sees at once that the digraph D_1 is a subgraph of the complete digraph D_2. The digraph D_2 is only possible if C^2 coalitions are allowed.

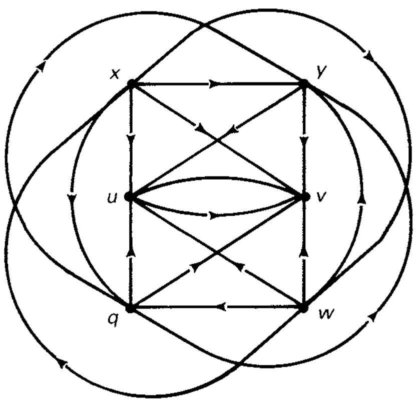

Fig. 2

It is important to realize that the fact that not all pairs of points are connected by arrows in a digraph D_h $(1 \leqq h < n)$ does not mean that there is any point which is not related to some other points by at least one arrow. For each outcome there are always some outcomes which are different from the former in at most h issues. Thus there are no isolated points in the digraphs D_h even if $h = 1$. In the language of graph theory all digraphs D_h are weakly connected digraphs (Harary, Norman, and Cartwright 1965, 51).

Assume now that a sequence of arrows exists which leads from one point, say x, to another, v (compare e.g., fig. 1). Then we say that v is R-reachable from x. Obviously, this is only the case if $x R \ldots R v$ is true. The set of all outcomes which are R-reachable from some x in the digraph D_h $(1 \leqq h \leqq n)$ will be called $F_h(x)$.[2] By definition we include x in this set. For the digraph D_1 of figure 1, we have $F_1(x) = \{x, u, q, y, v\}$, for the digraph D_2 of figure 2 $F_2(x) = \{x, u, q, y, v, w\}$.

We will further say that v is P-reachable from x if, as in figure 1

$$x\,P\;.\;.\;.\;.\;P\,v$$

is true. The set of all points in D_h which are P-reachable will be denoted by G_h (x). Obviously $G_h\,(x) \subseteq F_h\,(x)$.

We are now able to define the set of h-stable outcomes (points) and the corresponding subgraph $D_h{}^*$. Intuitively speaking this set $S_h \subseteq U$ comprises all $x \in U$ which are not P-reachable from any y outside S_h, $y \in U - S_h$, whereas y is P-reachable from at least one x belonging to S_h. More formally, for $x, y \in U$

$$U - S_h = \{y | \exists x \colon y \in G_h\,(x),\; x \notin G_h\,(y)\},$$

S_h is the complementary set to $U - S_h$. S_h may be called the general Condorcet set[3] in the case that no coalitions over more than h issues are allowed or considered.

Note that all $y \in U - S_h$ are unstable, since they are directly or indirectly dominated via P-relations by at least one $x \in S_h$. By contrast any $x \in S_h$ can at most be directly or indirectly dominated by other $z \in S_h$. S_h is thus the set of outcomes directly or indirectly dominating all other outcomes, if no k-issue coalitions $(h < k)$ are allowed.

The formal definition implies that no $z \in S_h$ is P-reachable from any $y \in U - S_h$. For assume that this were not true, then $z \in G_h\,(y)$. Moreover, we have from the definition $y \in G_h\,(x)$ and $x \notin G_h\,(y)$. Thus $z \in G_h\,(x)$. Now, if $x \in G_h\,(z)$ were valid then $x \in G_h\,(y)$, contrary to assumption. But if $x \notin G_h$ (z), then $z \in U - S_h$, since $z \in G_h\,(x)$, again contrary to assumption.

In figure 1 $S_h = \{x, w\}$, for $h = 1$. In figure 2 $S_h = \{w\}$ for $h = n = 2$. Finally in figure 3 $S_h = \{a, b, c, d\}$. Here h is not specified, but $h < n$, for the h-digraph is not complete.

The subgraph of the digraph D_h corresponding to S_h is the digraph comprising as points all elements of S_h and all directed lines (arrows) connecting them directly. It will be called digraph $D_h{}^*$.

c. The Formation of k-Issue Coalitions as a Reason for Intransitive or Cyclical Social Preferences

In this section we intend to prove some results, of which the logrolling theorem for separable individual preferences and the paradox of liberalism are just special cases (see sec. 2).

We begin by demonstrating the validity of the following.

LEMMA 1. *Each element of* $\mathrm{S_h}$ *is either a member of a social preference or indifference cycle within* $\mathrm{D_h}^*$ *or is an isolated point of the digraph* $\mathrm{D_h}^*$ *representing the elements of* $\mathrm{S_h}$ *and the* R*-relations between* *them. For* h = n *all elements of* S_h *are members of social preference or indifference cycles containing all other members of* S_h.

The proof follows directly from the definition of $U - S_h$. Assume $x, z \in S_h$. Then, if $x \in G_h(z)$ we must also have $z \in G_h(x)$. Otherwise $x \in U - S_h$, contrary to assumption. Further, if $x \notin G_h(z)$ for all $z \in S_h$ excluding x, then all $z \notin G_h(x)$. Otherwise, $z \in U - S_h$, contrary to assumption.

Thus either if $x \in G_h(z)$ $xP \ldots Pz$ or $xI \ldots Iz$ must hold with all outcomes in the chain belonging to S_h for some z; or if $x \notin G_h(z)$ x must be an isolated point in $D_h{}^*$.

Finally, if $h = n$ we have a complete digraph. Thus, we have at least one arrow connecting each pair of points, including $x, z \in S_n$. There is thus at least one arrow between all elements of S_n. But then $x \in G_n(z)$ and $z \in G_n(x)$ or $xI \ldots Iz$ from the above argument and thus x and z are members of a social preference or indifference cycle. This completes the proof.

We now proceed to prove:

THEOREM 1. *Assume that a nonempty set* $S_h \neq U$ *of* h-*stable outcomes exists and that* y, v, w $\in$ U *are outcomes such that* y *differs from* x $\in S_h$ *and* v, w *differ from each other in exactly* k *issues* ($1 \leqq h < k \leqq n$). *Then at least one cycle of intransitive social preferences, including* x $\in S_h$ *and* y $\in U - S_h$, *exists if*

a) for x $\in S_h$ *and* y $\in (U - S_h) \cap F_h(x)$, *a* C^k *coalition exists who prefers* y *to* x, *such that* y P x;

or if

b) for x $\in S_h$ *with* y, v $\notin F_h(x)$ *and* w $\in F_h(x)$, y $\in F_h(v)$, *two* C^k *coalitions exist, namely* ${}^1C^k$ *who prefer* y *to* x *so that* y Px, *and* ${}^2C^k$ *who prefer* w *to* v *so that* w Pv.

Let us observe that the social intransitivities mentioned in the theorem are a consequence of the possibility and usefulness to form winning coalitions over k issues, i.e., of an agreement concluded by k one-issue coalitions C_i to coordinate their decisions, an agreement which would be favorable to all their members.

The proof of theorem 1 is trivial. Let us turn first to part a. Since $y \in F_h(x)$ and $y\, Px$, we have

$$y\, P\, x\, R \ldots R\, z\, R \ldots R\, y.$$

Note that $z \in S_h$ is possible.

Next we prove part b. Here $y \in F_h(v)$, $y\, P\, x$, $w\, P\, v$ and $w = F_h(x)$. Thus $y\, P\, x\, R \ldots R\, w\, P\, v\, R \ldots R\, y$.

Moreover, since $y \notin F_h(x)$ two coalitions ${}^1C^k$ and ${}^2C^k$ are necessary to bring about intransitivity of group preferences involving x and y.

It is important to realize that in all cases in which $F_h(x) = U$, i.e., in which the digraph $D_h{}^*$ has no isolated subgraphs, only part a of theorem 1 is applicable. This includes the special case in which x is the only element of S_h. We may call it the general Condorcet solution for the case that only h-issue coalitions are allowed. It is obvious, moreover, that in this case cyclical social preferences are present, whenever it is profitable for a C^k coalition to form an agreement to realize $y \neq x$, $y \in U$, instead of x. For if x is the only element of $S_h \subset U$ then $y \in G_h(x)$ for $y \in U - S_h$.

Let us illustrate theorem 1 by looking at the digraphs of figures 1 and 2. We recall that figure 1 is a digraph D_1 which is a subgraph of the complete digraph D_2 of figure 2. $S_1 = \{x, w\}$, $w \notin F_1(x)$ in figure 1. Allowing C^2 coalitions gives us for adequate individual preference orderings and assignments of rights figure 2. Here we observe that a two-issue coalition ${}^1C^2$ exists such that $v\,P\,x$. Further $v \in F_1(x)$. Thus as asserted in theorem 1a we have:

$$v\ P\ x\ P\ y\ P\ v.$$

Next look at figures 3 and 4. Figure 3 shows the digraph D_h, which is a subgraph of the digraph D_k of figure 4. Since the latter digraph is not complete $k < n$. The possibility to form $(h + 1)=$, $(h + 2)=$, . . . , k-issue coalitions has led to two new arrows in figure 4 as compared to figure 3, namely from j to b, such that $j\,P\,b$, and from c to f, such that $c\,P\,f$. Since $j \in F_h(f)$ and $c \in S_h \cap F_h(b)$ we have, as asserted in theorem 1b:

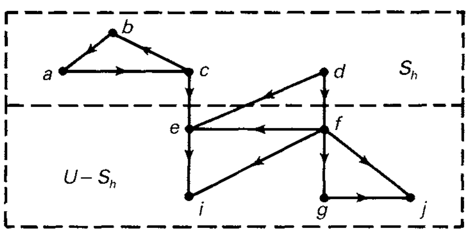

Fig. 3

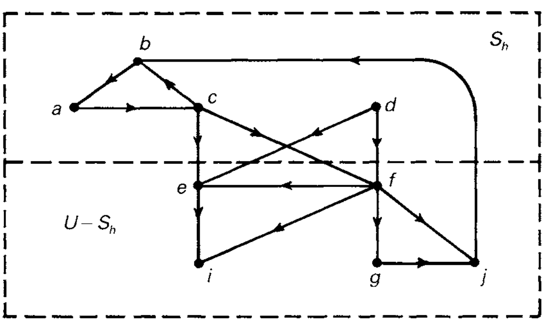

Fig. 4

$$j\ P\ b\ P\ a\ P\ c\ P\ f\ P\ g\ P\ j.$$

To conclude this section we turn to prove

THEOREM 2. *If the digraph* $\mathrm{D_h}$ *describing the situation in the absence of* (h + 1)=, (h + 2)=, . . . , k-*issue coalitions does not contain any social preference cycles, but if this is the case for the digraph* $\mathrm{D_k}$, *then this is a consequence of the formation of one or more* (h + 1)=, (h + 2)=, . . . , k-*issue winning coalitions benefiting all of their members.*

Again the proof is trivial. The only difference between the digraph D_k and its subgraph, the digraph D_h, is the fact that the digraph D_k contains one or more additional arrows between outcomes which are not connected in the digraph D_h. These arrows are in our interpretation of the digraphs the result of the possibility to form winning coalitions over more than h issues, or—in the case of social indifference—the impossibility to form such coalitions, i.e., the fact that blocking coalitions are possible. But social preference cycles as against intransitive social preferences can only be the result of winning but not of blocking coalitions over more than h issues. This completes the proof.

2. Interpretation of Results[4]

a. The Paradox of Liberalism

It is important to get an intuitive feeling of the meaning of the above results. We begin by showing with the help of an example that the paradox of a Paretian liberal found by Sen (1970) is just a special case of theorem 1*a*. In doing so we will follow the idea that under liberalism individuals have the right to decide certain *issues* for society.

In the example let us consider a society of three individuals $V = \{1, 2, 3\}$, who have to decide three issues M_i $(i = 1, 2, 3)$, each with two alternatives: $M_i = \{x_i, \bar{x}_i\}$. Each individual has the right to decide one issue, namely individual i issue M_i: $V_i = \{i\} \subset V$, $(i = 1, 2, 3)$. Obviously $U = \{x_1x_2x_3),$ $(\bar{x}_1x_2x_3)$, $(x_1\bar{x}_2x_3)$, $(x_1x_2\bar{x}_3)$, $(\bar{x}_1\bar{x}_2x_3)$ $(\bar{x}_1x_2\bar{x}_3)$, $(x_1\bar{x}_2\bar{x}_3)$, $(\bar{x}_1\bar{x}_2\bar{x}_3)\}$. Individuals have the following strong, complete, and transitive preference orderings.

TABLE 1

1	2	3
$\bar{x}_1x_2\bar{x}_3$	$\bar{x}_1\bar{x}_2\bar{x}_3$	$\bar{x}_1\bar{x}_2x_3$
$x_1\bar{x}_2\bar{x}_3$	$\bar{x}_1x_2\bar{x}_3$	$\bar{x}_1\bar{x}_2\bar{x}_3$
$\bar{x}_1\bar{x}_2\bar{x}_3$	$\bar{x}_1x_2x_3$	$x_1x_2x_3$
$\bar{x}_1\bar{x}_2x_3$	$x_1x_2x_3$	$\bar{x}_1x_2x_3$
$x_1x_2x_3$	$x_1\bar{x}_2x_3$	$\bar{x}_1x_2\bar{x}_3$
$x_1\bar{x}_2x_3$	$x_1x_2\bar{x}_3$	$x_1x_2\bar{x}_3$
$\bar{x}_1x_2x_3$	$x_1\bar{x}_2\bar{x}_3$	$x_1\bar{x}_2\bar{x}_3$
$x_1x_2\bar{x}_3$	$\bar{x}_1\bar{x}_2x_3$	$x_1\bar{x}_2x_3$

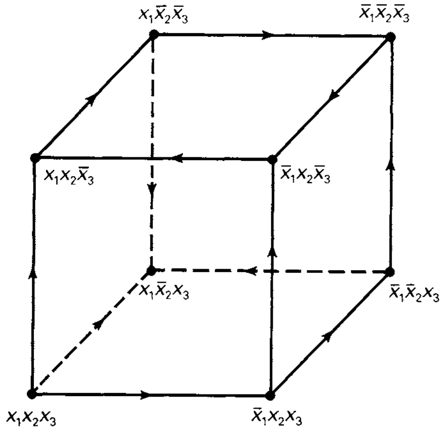

Fig. 5

The resulting digraph D_1 is given by figure 5. The digraph has been drawn in three dimensions to depict the three issues each with two alternatives.

The digraph D_1 contains only one one-issue stable solution $S_1 = \{x_1x_2x_3\}$. Consequently, $(\bar{x}_1\bar{x}_2\bar{x}_3) \in F_1\ (x_lx_2x_3)$. If we now consider the possibility of two- and three-issue coalitions, we find that $C^3 = \{1, 2, 3\}$ prefer $(\bar{x}_1\bar{x}_2\bar{x}_3)$ to $(x_1x_2x_3)$: $(\bar{x}_1\bar{x}_2\bar{x}_3)\ P\ (x_1x_2x_3)$. Thus an arrow has to be drawn from $(\bar{x}_1\bar{x}_2\bar{x}_3)$ to $(x_1x_2x_3)$. As a consequence we have, e.g.,

$$(\bar{x}_1\bar{x}_2\bar{x}_3)\ P\ (x_1x_2x_3)\ P\ (\bar{x}_1x_2x_3)\ P\ (\bar{x}_1\bar{x}_2x_3)\ P\ (\bar{x}_1\bar{x}_2\bar{x}_3).$$

Now realize that a three-issue coalition has to comprise with liberalism all three members of society. Thus $(\bar{x}_1\bar{x}_2\bar{x}_3)$ is Pareto-better than $(x_1x_2x_3)$. It follows that the cycle can be interpreted as a contradiction between pure liberalism, where each individual would decide his own affairs independently, and which would give $(x_1x_2x_3)$ as a solution, and the Pareto principle.

It is obvious, however, that this interpretation is somewhat doubtful. If individual i has the right under liberalism to decide issue M_i, why should he not have the right to make agreements with the other individuals as to their decisions if this is preferable? But then they can agree to decide in favour of the Pareto-optimal outcome $(\bar{x}_1\bar{x}_2\bar{x}_3)$ as against $(x_1x_2x_3)$. The real problem is, therefore, why the individuals either do not enter into such an agreement or why they violate it. In our example it is obviously to the advantage of 1 and (or) of 3 to break the agreement in the hope that the other may keep it or in the fear that the other may break it if one honours it oneself. For $(x_1\bar{x}_2\bar{x}_3)\ P_1\ (\bar{x}_1\bar{x}_2\bar{x}_3)$, $(\bar{x}_1\bar{x}_2x_3)\ P_3\ (\bar{x}_1\bar{x}_2\bar{x}_3)$ and $(\bar{x}_1\bar{x}_2\bar{x}_3)\ P_1\ (\bar{x}_1\bar{x}_2x_3)$, $(\bar{x}_1\bar{x}_2\bar{x}_3)\ P_3\ (x_1\bar{x}_2\bar{x}_3)$. Even 2 will be reluctant to conclude and to keep the agreement, since he has to fear its violation by the others, for $(x_1x_2x_3)\ P_2\ (x_1\bar{x}_2\bar{x}_3)\ P_2\ (\bar{x}_1\bar{x}_2x_3)$. All possible participants find themselves thus in a dilemma, which is certainly related to the allocation of rights to make decisions, but which is in no way a special trait of liberalism. On the contrary, as we could suspect from our theorems, this kind of dilemma is a general problem of a decentralized social system and may happen with any kind of such a social organization (see sec. 4). This will become clearer when we turn to the example of logrolling in the next section.

Concluding this section let us mention that F_1 $(x_1x_2x_3) = U$. Since, moreover, $S_1 = \{x_1x_2x_3\}$, theorem 1*a* applies, if any C^k $(k > 1)$ coalition is present. This is the case in the present example.

b. Simple Majority Logrolling

Let us again consider a society of three individuals $V = \{1, 2, 3\}$, who have to decide three issues M_i $(i = 1, 2, 3)$, each with two alternatives: $M_i = \{x_i, \bar{x}_i\}$. But now all individuals together have the right to decide each issue M_i, i.e., $V_i = V$ $(i = 1, 2, 3)$, by simple majority voting. The individual preferences are presented in table 2 and the resulting social preferences are again given by figure 5.

In this example, too, $S_1 = \{x_1x_2x_3\}$ and F_1 $(x_1x_2x_3) = U$. Thus cyclical group preferences result, since $C^3 = \{1, 2, 3\}$ prefer $(\bar{x}_1\bar{x}_2\bar{x}_3)$ to $(x_1x_2x_3)$. As a consequence we could speak of a "contradiction of the Pareto principle and of majority voting of all members of society", if separate votes would be taken on the three issues and if no agreement among them concerning their voting behavior were allowed. Thus Sen's paradox is not a special feature of liberalism. All members of society prefer $(\bar{x}_1\bar{x}_2\bar{x}_3)$ to $(x_1x_2x_3)$. But without logrolling $(x_1x_2x_3)$, which is not Pareto-optimal, will be the outcome.

On the other hand, as in the example given in section 2a for liberalism, it is not at all sure that a logrolling agreement will be concluded or kept if it has been concluded. For majority $C_1{}^2 = \{1, 2\}$ prefer $(x_1\bar{x}_2\bar{x}_3)$ and majority $C_2{}^2 = \{2, 3\}$ prefer $(\bar{x}_1\bar{x}_2x_3)$ to $(\bar{x}_1\bar{x}_2\bar{x}_3)$. Thus voters 3 and 1 cannot be sure that the voting agreement will be kept, and that not $(x_1\bar{x}_2\bar{x}_3)$ or $(\bar{x}_1\bar{x}_2x_3)$ will be the outcome instead of $(\bar{x}_1\bar{x}_2\bar{x}_3)$, if one of them sticks to the voting agreement and the others do not. But $(x_1x_2x_3)$ P_3 $(x_1\bar{x}_2\bar{x}_3)$ and $(x_1x_2x_3)$ P_1 $(\bar{x}_1\bar{x}_2x_3)$. This dilemma is, of course, nothing else than the cyclical group preferences present.

TABLE 2

1	2	3
$x_1\bar{x}_2\bar{x}_3$	$x_1x_2\bar{x}_3$	$\bar{x}_1x_2x_3$
$\bar{x}_1\bar{x}_2\bar{x}_3$	$x_1\bar{x}_2\bar{x}_3$	$\bar{x}_1\bar{x}_2x_3$
$x_1x_2x_3$	$\bar{x}_1\bar{x}_2x_3$	$\bar{x}_1\bar{x}_2\bar{x}_3$
$\bar{x}_1x_2x_3$	$\bar{x}_1\bar{x}_2\bar{x}_3$	$\bar{x}_1x_2\bar{x}_3$
$\bar{x}_1\bar{x}_2x_3$	$\bar{x}_1x_2\bar{x}_3$	$x_1\bar{x}_2x_3$
$\bar{x}_1x_2\bar{x}_3$	$x_1x_2x_3$	$x_1x_2x_3$
$x_1x_2\bar{x}_3$	$\bar{x}_1x_2x_3$	$x_1x_2\bar{x}_3$
$x_1\bar{x}_2x_3$	$x_1\bar{x}_2x_3$	$x_1\bar{x}_2\bar{x}_3$

c. Separable Individual Preferences

In the introductory section we mentioned that the logrolling theorem, which is a special case of theorems 1 and 2, has been derived with the help of the assumption of separable individual preferences in earlier articles. It is therefore useful to make clear the role of this assumption in these proofs. We do so by proving the following.

> THEOREM 3. *If separable individual preferences are present then the digraph* $\mathrm{D_h}^*$ *does not contain any subgraphs which are not* R*-reachable from each other, i.e., digraph* $\mathrm{D_h}^*$ *is a strongly connected digraph.*[5]

Before proving the theorem, let us shortly discuss its meaning. If the digraph D_h^* has no isolated subgraphs, then all points (outcomes) belonging to it have to be members of preference or indifference cycles within the digraph D_h^*. This follows directly from Lemma 1. Because of this only part *a* of theorem 1 and not part *b* is applicable in the case of separable individual preferences. We can, therefore, conclude that whenever separable individual preferences are present, and if a nonempty digraph D_h^* exists, intransitive social preferences will result including at least one member of $U - S_h$ and all members of S_h, if a C^k coalition exists who prefer one outcome $y \in U - S_h$ to another one $x \in S_h$. For the special case of simple majority voting on all issues by all members of society this means that a logrolling situation in which a majority prefer some $y \in U - S_h$ to some $x \in S_h$ implies intransitive group preferences. This allows, of course, the possibility of cyclical group preferences. Note that the consequences just mentioned are not confined to majority voting. As already realized by Schwartz (1977), they are valid for all the combinations of allocations of rights and decision rules which we have allowed in section 1a.

We turn now to the proof of theorem 3 and define first separable individual preferences.

Let $y, y^*, z, z^* \in U$ be such that

$$y = (y_1, y_2, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)$$
$$y^* = (y_1, y_2, \ldots, y_{i-1}, y_i^*, y_{i+1}, \ldots, y_n)$$

and

$$z = (z_1, z_2, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_n)$$
$$z^* = (z_1, z_2, \ldots, z_{i-1}, y_i^*, z_{i+1}, \ldots, z_n).$$

Then separable individual preferences are present, if $y\, R_j\, y^*$ implies $z\, R_j\, z^*$, $j \in V$ and for all $\mathrm{y_p}$, z_p with $q \neq i$.

Next let us define $y^a \in U$ as those outcomes which are identical with y in the first α issues and with x in the last $n - \alpha$ issues. Therefore

$$y^1 \equiv (y_1, x_2, \ldots, x_n)$$
$$y_2 \equiv (y_1, y_2, x_3, \ldots, x_n)$$
$$\vdots \qquad \vdots$$
$$y^{k-1} \equiv (y_1, y_2, \ldots, y_{k-1}, x_k, \ldots, x_n).$$

We state first that theorem 3 is trivally true, if S_h contains only one element. Next, all outcomes in S_h differing in not more than h issues are obviously connected by at least one arrow in $D_h{}^*$, since 1-, 2-, . . . , h-issue coalitions are allowed.

Consider now two outcomes x, $y \in S_h$, which are different in k components ($1 \leqq h < k \leqq n$). Without loss of generality we can assume that x and y differ in the first k outcomes. Thus, let

$$x \equiv (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n)$$
$$y \equiv (y_1, y_2, \ldots, y_k, x_{k+1}, \ldots, x_n)$$

We assume next that x, $y \in S_1$ are any points belonging to digraph $D_1{}^*$. If we can show, as will be done below, that the assumption of separable individual preferences implies $y = F_1(x)$, then $x = F_1(y)$ must be true, too, because of Lemma 1. As a consequence no x, $y \in S_1$ belong to isolated subgraphs of $D_1{}^*$ and theorem 3 is valid for the case $h = 1$. Finally we will demonstrate that the result can be generalized to the case $h > 1$, i.e., for the digraph $D_h{}^*$.

Let us begin by proving $y = F_1(x)$. We observe first that if for all members of some C_i coalition $y\, R_j\, y^*$ is valid then $z\, Rj\, z^*$ is true for the same members because of the assumption of separable individual preferences. But since C_i is a winning coalition for issue M_i it follows at once that we get $y\, R\, y^*$ and $z\, R\, z^*$ for society. The former relation implies the latter.

Now consider ${}^\alpha y \in U$, where

$${}^\alpha y \equiv (x_1, x_2, \ldots, x_{\alpha-1}, y_\alpha, x_{\alpha+1}, \ldots, x_n)\ (\alpha = 1, 2, \ldots, n),$$

and note that ${}^\alpha y$ differs from x only in one component, namely component α. Consequently there must be an arrow between x and ${}^\alpha y$. From this and since $x \in S_1$, if ${}^\alpha y \in U - S_1$ then

$$x\, R\, {}^\alpha y. \tag{1}$$

Further, if ${}^\alpha y \in S_1$, either the same relation holds or if ${}^\alpha y\, R\, x$,

$$x\, R \ldots R\, v\, R\, w\, R \ldots R\, {}^\alpha y, \tag{2}$$

because of Lemma 1. Here all elements belonging to this indirect relation between x and y are elements of S_1, e.g. v, $w \in S_1$. It is important to realize that v and w can only be different in one issue, since they belong to S_1.

From (1) we get for $\alpha = 1$

$$x \; R \; y^1. \tag{3}$$

Similarly from (2)

$$x \; R \; . \; . \; . \; R \; y^1 \text{ if } {}^1y \; R \; x. \tag{3a}$$

Next from (1) for $\alpha = 2$ we conclude by using the assumption of separable individual preferences:

$$y^1 \; R \; y^2. \tag{4}$$

Similarly from (2) for $\alpha = 2$ and using the same assumption,

$$y^1 \; R \; . \; . \; . \; R \; v^1 \; R \; w^1 \; . \; . \; . \; R \; y^2, \text{ if } {}^2y \; R \; x. \tag{4a}$$

This conclusion has to be explained in detail. First note that

$$\begin{aligned} v &\equiv (v_1, v_2, \ldots, v_n) \\ v^{\alpha - 1} &\equiv (y_1, y_2, \ldots, y_{\alpha - 1}, v_\alpha, \ldots, v_n) \\ w &\equiv (w_1, w_2, \ldots, w_n) \\ w^{\alpha - 1} &\equiv (y_1, y_2, \ldots, y_{\alpha - 1}, w_\alpha, \ldots, w_n). \end{aligned}$$

Since we consider the digraph D_1 it follows that w is different from v in only one component. The same is true for all neighboring outcomes in (2). Now by replacing the first $\alpha - 1$ components in v and w by $y_1, y_2, \ldots, y_{\alpha - 1}$ either leaves the component in which w and v differ unchanged in $w^{\alpha - 1}$ and $v^{\alpha - 1}$. Then $w^{\alpha - 1}$ and $v^{\alpha - 1}$ differ in the same single component as w and v and it follows from the assumption of separability that $v \; R \; w$ implies $v^{\alpha - 1} \; R \; w^{\alpha - 1}$. Or the only component in which w and v differ is replaced by the same y_i to get $w^{\alpha - 1}$ and $v^{\alpha - 1}$. Then $v^{\alpha - 1} = w^{\alpha - 1}$, $v^{\alpha - 1} \; I \; w^{\alpha - 1}$ and, therefore, $v^{\alpha - 1} \; R \; w^{\alpha - 1}$ holds.

As a consequence, we get from (2)

$$y^{\alpha - 1} \; R \; . \; . \; . \; R \; v^{\alpha - 1} \; R \; w^{\alpha - 1} \; R \; . \; . \; . \; R \; y^\alpha. \tag{2a}$$

Setting $\alpha = 2$, (4a) follows.

Moving on to $\alpha = 3$ we derive from (1) and (2a) respectively

$$y^2 \; R \; y^3 \tag{5}$$

or

$$y^2 \; R \; . \; . \; . \; R \; v^2 \; R \; w^2 \; R \; . \; . \; . \; R \; y^3 \tag{5a}$$

for $^3y\ R\ x$ etc. for $\alpha = 4, 5, \ldots$ until we finally get

$$y^{k-1}\ R\ y \tag{6}$$

or

$$y^{k-1}\ R \ldots R\ v^{k-1}\ R\ w^{k-1}\ R \ldots R\ y \text{ for } {}^{k-1}y\ R\ x. \tag{6a}$$

Putting our results together we have

$$x\ R \ldots . R\ y. \tag{7}$$

Thus $y \in F_1\ (x)$ and because of Lemma 1 $x \in F_1\ (y)$. x and y do not belong to isolated subgraphs of the digraph $D_1{}^*$.

Until now we have only shown that no pair of outcomes $x, y \in S_1$ belongs to isolated subgraphs of the digraph $D_1{}^*$. But is the same true for the digraph $D_h{}^*$ $(1 < h \leqq k)$? It is easily shown that this is in fact the case. We have just demonstrated that for each $y \in S_1$ we have $y \in F_1\ (x)$ for all $x \in S_1$. But then, following from the definition of h-stability for all $z \in U - S_1$ we have also $z \in F_1\ (x)$ for all $x \in S_1$. Thus each point (outcome) is R-reachable from x. But this cannot change if h-issue coalitions are allowed $(h > 1)$. For this results only in a number of new arrows between the points. Thus if $x, y \in S_1$ are also members of S_h then they are not isolated as before. And if for some $z \in U - S_1$ we have $z \in S_h$, then this can only be the case through a C^k coalition for whose members $z\ P_j\ y$ is valid for some $y \in S_1$. For since $z \in F_1\ (y)$ it is only possible that $z \in S_h$ according to the definition of h-stability, if $y \in F_h\ (z)$. But then $z \in F_h\ (x)$ for all $x \in S_h$, because $y \in F_1\ (x)$. This completes the proof of theorem 3.

3. Relation to Arrow's General Impossibility Theorem

It has been shown in section 1 that if intransitive social preferences are absent in the digraph D_1 but present in the digraph D_n, this is the consequence of the possibility of 2-, 3-, . . . , n-issue coalitions benefiting all of their members (see theorem 2). The presence of intransitive social preferences in the digraph D_n can be interpreted as a contradiction between the assumptions if transitivity of the complete weak social preference ordering represented by the digraph D_n is postulated.

Apart from the case of intransitive social preferences in the digraph D_1 our results can thus be looked at as a substantive interpretation and a proof of Arrow's General Impossibility Theorem (Arrow 1963), since the existence of a C^n coalition can be shown for all social organizations considered (see section 4). For it is obvious that, excluding dictatorship, all conditions postulated by Arrow are fulfilled by our assumptions, as will be demonstrated. But if this is true, then the impossibility of a nondictatorial transitive social preference ordering for all possible combinations of individual preference orderings is only a consequence of the possibility to form 2-, 3-, . . . , n-issue coalitions benefiting all of their members in a decentralized social system, i.e., of the possibility to conclude agreements among C_i coalitions furthering the interests of all of their members. But if such C^h ($h = 2, \ldots, n$) coalitions are not allowed, non–Pareto-optimal outcomes may result as has been shown for the examples of sections 2a and 2b. With this prohibition of C^h coalitions one could therefore speak of a contradiction between Pareto–optimality and liberalism as has been done by Sen. But obviously such a restriction is not a trait of liberalism alone, but more generally for all forms of decentralization.

Let us show now that Arrow's conditions are fulfilled by our assumptions. In doing so we will, however, rely on the somewhat changed formulation used by Luce and Raiffa (1957, 333–40) and adapt it to the notation used above. We shall denote by a *profile of preference orderings* for the individuals of society an m-tuple of orderings, $(R_1, R_2, \ldots, R_m)$, where R_j is the preference ordering for individual $j \in V$ for some $x, y \in U$.

We discuss first

CONDITION 1.

a) *The number of outcomes in* U *is greater than or equal to three.*
b) *The social preference ordering is defined for all possible profiles of individual orderings; it is a weak, complete and transitive ordering.*
c) *There are at least two individuals.*

Conditions 1*a* and 1*c* are obviously met by our assumptions. Moreover, we have put no restrictions on possible individual preference orderings. With all h-issue coalitions allowed ($1 \leqq h \leqq n$), i.e., for the digraph D_n, our assumptions imply a weak and complete social preference ordering. Thus condition 1*b* is fulfilled with the exception of transitivity of the social preference ordering. But since, as will be seen, conditions 2 to 4 below are implied by our assumptions, intransitivity is a possible consequence of nondictatorship. This is the only reason why we did not include transitivity among our assumptions. If we do so condition 1 is fulfilled and the contradiction proved by Arrow is present in our system.

Let us turn to the condition of the positive association of social and individual values.

CONDITION 2. *If the social preference relation asserts that* x *is preferred to* y *for a given profile of individual preferences, it shall assert the same when the profile is modified as follows:*

a) *The individual paired comparisons between outcomes other than* x *are not changed.*
b) *Each individual paired comparison between* x *and any other outcome either remains unchanged or it is modified in* x*'s favor.*

Condition 2 is obviously fulfilled by our assumptions of section 1, for $x\ P\ y$ is only dependent on the fact that $x\ P_j\ y$ is valid for all members $j \in C$ of the corresponding C coalition. And if some individual $s \notin C$ changes his preference ordering from $y\ R_s\ x$ to $x\ P_s\ y$ we get either a coalition $C' = C \cup \{s\}$ which is according to our assumption still a winning coalition, so that $x\ P\ y$ holds, or $s \notin C_i$ for all possible C_i out of which some C'' can be formed. Then the change of his preference ordering has no consequences at all. The same is true if several members of $V - C$ change their preference orderings.

Let us now discuss the condition of the independence of irrelevant alternatives.

CONDITION 3. *Let* U_1 *be any subset of outcomes in* U. *If a profile of orderings is modified in such a manner that each individual's paired comparisons among the outcomes in* U_1 *are left invariant, the social orderings resulting from the original and modified profiles of individual orderings should be identical for the outcomes in* U_1.

This condition is obviously fulfilled by the assumptions of section 1, for if $x\ P_j\ y$ for all $j \in C$, $x, y \in U_1$, with C being a winning coalition for x against y, we have $x\ P\ y$. And this relation is not changed if $v\ P_s\ w$ is changed into $w\ R_s\ v$ for some $s \in V$ and for some $v, w \in U - U_1$.

Next we turn to citizen's sovereignty.

CONDITION 4. *For each pair of* x *and* y, *there is some profile of individual orderings such that society prefers* x *to* y.

Condition 4 is also fulfilled by our assumptions. If x and y are different only in issue M_i, then there exists for all profiles of individual orderings referring to all individuals $s \in U - V_i$ a $L_i \subseteq V_i$ such that $x\ P\ y$ if for all $j \in L_i$ $x\ P_j\ y$. But if this is true then there must be, moreover, a C^h composed out of h C_i coalitions such that $x\ P\ y$, if x and y are different in h issues $(1 < h \leqq n)$, and if $x\ P_j\ y$, $j \in C^h$.

Finally we have the condition of nondictatorship.

CONDITION 5. *There is no individual with the property that whenever he prefers* x *to* y *(for any* x *and* y*) society does likewise, regardless of the preferences of other individuals.*

Condition 5 is not one of our assumptions. But it is easy to see that without it no contradiction among the conditions can result. Or put within the framework of our discussion no social intransitivity can be present. For according to our assumptions individual preference orderings are weak, complete, and transitive. Thus, if the preference orderings of one individual determine social preference relations, the social preference relation must be transitive, too. Thus we have to assume nondictatorship to get social intransitivity and a contradiction between the conditions.

4. Proof of the Possibility of Cyclical Social Preferences

We have still to show that our assumptions together with the assumptions of nondictatorship and transitivity of social preferences lead to a contradiction for all assignments of rights and decision rules considered. This will be done by showing that there exists at least one profile of individual preferences allowing the formation of $C_1, C_2, \ldots, C_n$ coalitions and a C^n coalition which imply cyclical social preferences, given the other assumptions.

Let us first state

THEOREM 4. *Assume that any assignment of rights and any decision rules allowed in section 1 are present and that there exists at least one collection of winning coalitions* $C_1, C_2, \ldots, C_n$ *such that*

$$C_1 \cap C_2 \cap \ldots \cap C_n = \varnothing. \tag{8}$$

Then there exists at least one profile of individual preferences which implies cyclical social preferences.

What is the meaning of the additional assumption? Obviously, if $C_1 \cap C_2 \cap \ldots \cap C_n \neq \varnothing$ would hold for all possible combinations of one-issue winning coalitions, there would be one or more individuals able to block all decisions favoring any outcome preferred by all other members of society to any other. These individuals would be members of a kind of veto-oligarchy, each of whose members were able to prevent any decision of society opposed to their interests. The additional assumption thus implies that there is at least one collection of winning coalitions without members who are "blocking dictators." Note that this assumption is weaker than nondictatorship which is implied by it.

To prove the theorem assume the following individual preferences:

$$\begin{aligned}
&xP_{j2}\, y \text{ for all } j_1 \in C_1,\\
&yP_{j_l}\, x \text{ for all } j_1 \in V - C_1;\\
&yP_{j2}\, u \text{ for all } j_2 \in C_2,\\
&uP_{j2}\, y \text{ for all } j_2 \in V - C_2:\\
&\vdots \qquad \vdots\\
&wP_{j_n}\, z \text{ for all } j_n \in C_n,\\
&zP_j\, w_n \text{ for all } j_n \in V - C_n.
\end{aligned} \tag{9}$$

Because of (9) we get

$$xPyPuP \ldots . PwPz. \tag{10}$$

Moreover, since because of (8) nobody can belong to all C_i ($i = 1, 2, \ldots, n$) in (9), everybody has to belong to at least one $V - C_i$. Thus

$$(V - C_1) \cup (V - C_2) \cup \ldots \cup (V - C_n) = V. \tag{11}$$

But from this and (9) it follows that for no $j_i \in V$

$$xR_{j_i}\, yR_{j_i}\, uR_{j_i} \ldots R_{j_i}\, w\, R_{j_i}z \tag{12}$$

is valid.

Now assume

$$zP_{j_i}\, x \text{ for all } j_i \in C^n \tag{13}$$

and thus

$$zPx. \tag{14}$$

Since (12) is not valid for any $j_i \in C^n \subseteq V$, (13) does not lead to a contradiction with the assumption of transitive individual preferences. Consequently C^n exists, since it could contain up to all members of society. It can be formed, e.g., as $V_1 \cup V_2 \cup \ldots \cup V_n$.

But from (10) and (14) we get cyclical social preferences. This completes the proof.

Finally let us show that Sen's result for liberalism can be generalized to the set of organizations of society considered and for which (8) is valid. For assume that

$$C^n = V. \tag{15}$$

Next let us follow the same procedure as in (9) to (14). Because of (10) x would be the outcome if no agreement to decide in favor of z among some $C_1', C_2', \ldots, C_n'$ forming C^n would be made. But clearly, because of (13) and (15) z is Pareto-better than x. Thus we may speak of a contradiction of the respective kind of organization of society and the Pareto principle.

Note that the case of liberalism with $C_i = V_i = \{j_i\}$, $j_i \in V$, $i = 1, 2, \ldots, n$ and $V_1 \cup V_2 \cup \ldots \cup V_n = V$ fulfills (8) and (15).

It is important to realize that if each $j_i \in V$ belongs to at least one V_i then assumption (8) implies $n \geqq m$ for some but not all kinds of organizations of society. This is true for the case of liberalism just stated, but it is also true, if $V_i = V$ for all i and if $m - 1$ members of society have to agree to get a decision in favor of one of two alternatives out of any issue.

On the other hand, with simple majority voting of all members of society on all issues $n \geqq 3$ is sufficient to bring about Sen's contradiction irrespective of the size of $m \geqq 3$ (see example in sec. 2).

NOTES

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1. Compare Harary, Norman, and Cartwright 1965.
2. See Harary, Norman, and Cartwright 1965, 85.
3. The expressions *Condorcet solution* and *Condorcet set* refer to majority voting. Here we use them with the adjective *general* since they refer to the very general form of decision making we consider.
4. More examples with different forms of social organizations can be found in Bernholz 1978.
5. A digraph is strongly connected if every two points are mutually R-reachable (Harary, Norman, and Cartwright 1965, 51).

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