Towards a Theory of Yes-No Voting

Roger L. Faith and James M. Buchanan

The formal theory of majority rule voting has dealt almost entirely with the unique selection of a single candidate(s) or motion(s) from a set of alternatives greater than two. The analysis presumes that there is only a single group or collective decision to be made, a single "election," or a single "proposition."[1] This orthodox conceptual setting for collective choice is necessary to generate the possibility of the cyclically rotating, and hence disequilibrium, set of outcomes on the one hand and for the median-voter–dominated equilibrium outcome on the other. If the number of alternatives in the choice set is limited to two, simple majority rule voting yields unambiguous results provided only that we assume an odd number of voters with each voter assumed to have strictly ordered preferences. In the orthodox voting model setting, few problems of analytical interest seem to arise in the single pairwise choice between two alternatives, for example, between approval and disapproval of a proposition.

When the voting population is presented with a whole set of independent propositions, however, each one of which is to be resolved by simple "yes-no," "up-down," or "approve-disapprove" majority voting, questions of considerable analytical interest do arise. To our knowledge no one has discussed or investigated the analogues and contrasts between yes-no voting and the conventional, multi-alternative, majority rule voting. The relative neglect of the properties of yes-no voting under majority rule is itself puzzling since many real-world collective choice institutions formally operate in this way. Examples that come to mind include zoning boards, referenda, and initiatives; in several states judges are reelected on a yes-no basis.

The Basic Model

The institutional setting we wish to analyze is one in which there are several propositions to be settled by simple majority rule. The propositions are characterized by dichotomous choice—yes or no—and counterproposals are not permitted.[2] The propositions are independent in the sense that none, some, or all may be approved as a result of the voting process. There is no technological or consumptive complementarity or mutual exclusivity among propositions. With respect to voters, we assume they vote without strategic considerations. Vote trading or other mechanisms by which differences in preference intensities are resolved are precluded here. Each voter makes his or

First published in *Public Choice* 37, no. 2 (1981): 231–45.

The Theory of Public Choice - II
James M. Buchanan and Robert D. Tollison, Editors
http://www.press.umich.edu/titleDetailDesc.do?id=7229
The University of Michigan Press, 2009.

her own subjective evaluation of each proposition being voted upon. Note that this assumption does not preclude representation, where an individual voter acts as an agent or ''representative'' of some constituency or even for some special interest group.

For purposes of exposition let us initially consider a board of zoning appeals made up of three members. Their job is to vote, yes or no, on separate requests for zoning variances. Several requests are allowed to accumulate before any voting occurs, say, before the regular monthly meeting of the board. Suppose that six different, and independent, requests for zoning variances are placed on the agenda facing the three-person board.

As a first step, assume that the individual evaluations (or subjective benefit-cost computations) of the different requests for variance are such that all board members ordinally rank the six requests in the same way. This does not mean that evaluations for all six proposals are identical over all three members in some cardinal sense—the precise benefit-cost ratios need not be equal. If such a level of ''objectivity'' could be reached, no decision problem would arise and all members would agree unanimously on the number and identity of proposals to be approved. Equivalence of ordinal rankings means only that if each member were asked to rank or array the six proposals according to his/her subjective estimates of benefits and costs, the three rankings would be the same. This assumption of equivalent ordinal ranking does not tell us anything about how many proposals are looked upon favorably or unfavorably by the individual voters. Two voters could have the same ranking with one voter approving all proposals while the other voter approves none.

A hypothetical set of ordinal rankings is illustrated in figure 1 where the rows represent the six requests, numbered 1 through 6, and the columns represent the three voters, *i*, *j*, and *k*. The vote of each member is either yes (Y) or no (N) on each request as it comes under consideration.

	i	*j*	*k*
1	Y	Y	Y
2	Y	Y	Y
3	Y	Y	N
4	Y	N	N
5	N	N	N
6	N	N	N

Fig. 1

Analogues and Contrasts to Conventional Majority-Rule Voting Models

The three board members, illustrated in figure 1, differ in the number of requests approved. Member *i* votes yes on two-thirds of the requests, *j* votes yes on one-half of the requests, and *k* votes yes on only one-third of the requests. Stated alternatively, the three voters or board members differ along what we might term the "proclivity to affirm" or "yea-saying" dimension.[3] If the six proposals are generically equivalent, e.g., all requests for zoning variances are to allow the development of separate and unrelated entrepreneurial projects, the underlying dimension might be classified on some scale of optimism-pessimism about the potential productivity of the projects. For other kinds of proposals, the underlying dimension might fall along ideological lines such as "liberal-conservative." Member *i* tends to evaluate benefits highly and costs minimally, while member *k* estimates costs highly and benefits minimally. Member *j* falls somewhere in between.

Under majority rule voting, requests 1, 2, and 3 will be approved while requests 4 through 6 will fail to secure the requisite majority of yes votes. This set of outcomes, not coincidentally, is precisely that set of outcomes desired by member *j*. In a very real sense, *j*'s decision-making status is the same as that of the median voter in conventional voting models where a single alternative is to be selected from a set of mutually exclusive alternatives and where individual preferences are single-peaked. In yes-no voting the analogue to the single-peakedness is the equivalence of the ordinal rankings of the separate propositions across all voters. The board member whose position is median on the "proclivity to affirm" dimension tends to dominate the results. The difference between this median voter and the conventional one is that here *median* is not defined with reference to the evaluation of a set of separate mutually exclusive alternatives from which one is to be selected. Rather *median* is defined with respect to the relative frequency of favorable votes cast in a set of independent proposals. The relationship between the median-voter construction in conventional models and that introduced here can be illustrated by a modification of our example. Consider a school board facing separate proposals for spending on six new schools, each costing $1 million. Voter *i*, in our construction, will approve four such proposals, voter *j* three, and voter *k* two. But note that this example might be translated into a single-election, conventional model by introducing the total budget dimension, in which case voter *j*, who prefers the $3 million outlay, will dominate the result.

It is, however, highly restrictive to assume that the ordinal rankings of the separate projects are identical over all board members, perhaps even more restrictive than to assume single-peakedness in preferences for all voters in the conventional majority rule voting models. Since the benefit-cost calculus must in any case be highly subjective, it seems apparent that the ordinal rankings would probably differ across voters or board members. How would this affect our median-voter result? As we shall see shortly, interpersonal differences in ordinal rankings need not make our "proclivity to affirm" dimension lose all of its descriptive or predictive value.

Under nonidentical ordinal rankings, the comparisons of our model of yes-no voting with the model of majority rule voting in the absence of single-peakedness are both interesting and complex. A characteristic feature of the latter voting model is the absence of any stable equilibrium. In contrast, our model of yes-no voting always yields a stable and unique voting outcome. Despite variation in the ordinal rankings across voters, one need only count up the number of yes votes cast on each proposal to see if a proposition is or is not approved. There will always be a definitive number of proposals approved; the voting process terminates. But will the median voter, the person who approves the median number of proposals, still dominate the proceedings? In general, the answer is negative. That is, there is an analogue between our model of yes-no voting on several issues and the cyclical majority model in terms of the loss of direct correspondence between the preferences of the median voter and the actual set of emergent outcomes.

Despite the fact that there will be a unique set of outcomes given *any* set of individual evaluations, one cannot predict, in advance, that these outcomes will correspond to those desired by the board member who is median in terms of our "proclivity to affirm" scale. However, the median voter does not lose all correspondence with the voting outcome, at least for a variety of different distributions of "proclivities for yea-saying" among voters. In the conventional model with a majority rule cycle, by contrast, the voter with median preferences is no more likely to get his preferences satisfied than any other voter within the range of cyclical results. Median preferences are, of course, hard to define in the cyclical or rotating majority case, but for the sake of comparison we may say that along some dimension measured independently of the voting process, the median voter is defined as the one whose *first preference* is median with respect to the first preference of other voters. As the voting outcomes rotate among the possible outcomes, no voter is more successful than others in the relevant set of achieving his/her most preferred outcome.

Yes-No Voting Power Indices

Having claimed that in yes-no voting the person with median "proclivities to affirm" has, in most cases, a greater likelihood of having his/her preferences satisfied than his/her counterparts, the question of interest is under what conditions is our claim valid? How likely is the median voter to have his/her preferences satisfied vis-à-vis other nonmedian voters?[4]

As mentioned above, given any pattern of ordinal rankings or propositions by the members of the voting group, a definitive set of outcomes results. And by association, some members' own preferences over the defined set of propositions correspond more closely with the set of outcomes than others. There is, of course, no reason to assume that the voter with the greater number of issues resolved in his/her favor is more "satisfied" in some utility sense. Such an implication would, of course, require an assumption about the intensities and interpersonal comparabilities of individual preferences.

We can, however, in some expected sense identify that voter who will have the greatest number of issues resolved in accord with his/her own preferences. (We shall, largely for purposes of expositional economy, refer to the relative positions of voters along such an "ability to get preferences satisfied" scale as "power," but the restricted meaning of this term should be kept in mind.) Consider again the example illustrated in figure 1. In order to ascertain voter *i*'s chance of having his/her way on each of the six issues, we assume, without loss of generality, that the six issues are numbered in accordance with *i*'s subjective benefit-cost ratios with issues 1 through 4 having benefit-cost ratios greater than one and issues 5 and 6 ratios less than one. We do not specify the ordinal rankings for voters *j* and *k*, but we retain the "proclivity to affirm" parameters for these two voters at one-half and one-third. Given any conceivable evaluation of the propositions by voters *j* and *k*, the probability of any given proposal securing majority approval *given that* i *votes yes* is .67. The probability of a given issue not passing *given that* i *votes no* is .83.[5] The *power index* for voter *i* can now be defined as the number of times *i* votes yes multiplied by the probability of majority approval plus the number of times *i* votes no multiplied by the probability of majority disapproval. Hence, in the current example the power index for *i* is $(4 \times .67) + (2 \times .83) = 4.34$. Similar calculations for voters *j* and *k* yield power indices of 4.67 and 4.34 respectively. Note that, as claimed earlier, voter *j*, the person with the median proclivity to approve propositions, has the highest power index.[6]

The most straightforward interpretation of the power index is that it measures the expected number of propositions to be resolved in a particular voter's favor. That is, over repeated votes taken on the same set of six propositions by the same three voters with fixed preferences, voters *i*, *j* and *k* can expect to have 4.34, 4.67, and 4.34 propositions, respectively, resolved in accord with their preferences. In order to compare and contrast indices when the number of proposals vary, we can "normalize" by dividing the power indices by the number of propositions yielding power indices of .72, .78, and .72 respectively. These numbers are interpreted as the expected fraction of group decisions resolved in the three voters' respective favors.

To see how the power index changes and the effect on the median voter's influence when the parameters of the model change we shall initially hold the number of voters fixed at three. Let N equal the number of propositions under consideration, and V_i, V_j and V_k equal the number of yes votes that voters *i*, *j*, and *k*, respectively, are expected to cast when presented with N independent propositions. Thus, V_i/N equals the probability of voter *i* voting yes on a given proposal. Further, assume that $V_i > V_j > V_k$. Generalizing the formulation presented above for the simple case, the power indices (PI) of the three voters can be written:

$$PI(i) = \frac{1}{N}[V_iV_j + V_iV_k - V_jV_k + N(N - V_i)]$$

$$PI(j) = \frac{1}{N}[V_iV_j + V_jV_k - V_iV_k + N(N - V_j)] \qquad (1)$$

$$PI(k) = \frac{1}{N}[V_iV_k + V_jV_k - V_iV_j + N(N - V_k)].$$

It follows that:

$$PI(j) - PI(k) = (2V_i - N)(V_j - V_k)/N \tag{2}$$

and that given $V_i > V_j > V_k$,

$$PI(j) - PI(k)\{\gtreqless\}0 \text{ if } V_i\{\gtreqless\}N/2. \tag{3}$$

Further, it follows that:

$$PI(j) - PI(i)\{\gtreqless\}0 \text{ if } V_k\{\gtreqless\}N/2. \tag{4}$$

From conditions (3) and (4) we see that the median voter has the highest power index if

$$\frac{V_i}{N} > \frac{1}{2} > \frac{V_k}{N}.^{7} \tag{5}$$

Thus, in the three-voter case, the power of the median voter depends solely on the ''proclivities to affirm'' of the two extreme voters. The example illustrated in figure 1 where $V_i/N = .67$ and $V_k/N = .33$ satisfies the above condition and, as we saw, voter j has the highest power index. It also follows from (3) and (4) that if either of the inequalities in (5) is an equality the median voter shares the highest power index with that voter whose ''proclivity to affirm'' is not equal to one-half. For example, if $V_i = 3$, $V_j = 2$, and $V_k = 1$, implying that $V_i/N = \frac{1}{2}$, the power indices of i, j, and k are 4.16, 4.83, and 4.83, respectively; while if $V_i = 5$, $V_j = 4$, and $V_k = 3$, implying that $V_k/N = \frac{1}{2}$, the power indices are 4.83, 4.83, and 4.16 respectively. Finally, it follows from (3) and (4) that if all voters have ''proclivities to affirm''greater (less) than one-half, the voter with the highest (lowest) proclivity has the highest power index. These relationships hold regardless of the absolute difference between any two voters' proclivity to ''yea-say.''

Before illustrating the effect of adding more voters, some simple comparative statics can be derived for voter i from conditions (1).[8] For greater generality we make no assumptions regarding the relative magnitudes of V_i, V_j and V_k.

$$\frac{\Delta PI(i)}{\Delta V_i} = \frac{V_j}{N} + \frac{V_k}{N} - 1, \tag{6}$$

$$\frac{\Delta PI(i)}{\Delta V_j} = \frac{V_i}{N} - \frac{V_k}{N}, \tag{7}$$

$$\frac{\Delta PI(i)}{\Delta N} = \frac{V_jV_k - V_i(V_j + V_k)}{N^2 + N} + 1 \tag{8}$$

Equation (6) states an increase in one's own "proclivity to affirm" will increase one's power index if the sum of the affirming proclivities of the remaining voters is greater than one and contrariwise. If the sum of the other two voters' proclivities to affirm is greater than one, then, on average, these voters prefer that a proposal will pass. Thus, an increase in i's desire to see more propositions passed will be reinforced with an expectation of favorable votes from the remaining voters, increasing i's power index. On the other hand if the voters on average tend to disapprove propositions, an increase in voter i's approval proclivity is a move away from the majority position and hence will reduce his power index. Equation (7) states that an increase in the "proclivity to affirm" of another voter increases (decreases) i's power index if the remaining voter is less (more) likely to vote affirmatively than the voter i. If voter i is more prone to voting yes than voter k, then i's power index will rise with an increase in j's proclivity to affirm which reinforces i's preferences relative to k's. If $V_k > V_i$ then an increase in V_j reinforces voter k's preferences relative to i's, and i's power index will fall. Finally, equation (8) states that an increase in the number of propositions increases or decreases one's power index depending upon the distribution of proclivities among the three voters.

Consider now an increase in the number of voters from three to five. The power index for each of the five voters is calculated in a similar manner as for three voters. In this expanded case, how does the median voter's power index compare to that of the remaining four voters?

For the sake of brevity, let us compare the median voter, voter k in the five-voter model, with the two voters who tend to be less prone to vote yes than k, voters l and m. In table 1, we illustrate a hypothetical case of five voters voting on ten propositions, their respective voting proclivities and resulting power indices. Note that once again the median voter is, in one sense, more "powerful," and because of the symmetric distribution of probabilities, the pairwise extreme voters on either side of the median voter have identical indices. Letting V_i, V_j, V_k, V_l, and V_m represent the expected number of yes votes to be cast on a set of N propositions for voters i, j, k, l, and m respectively, it follows that:

$$PI(k)\{\gtreqless\}PI(l) \text{ if } 2N(V_iV_j + V_lV_m + V_jV_m)\{\gtreqless\}N^3 + 4V_iV_jV_m, \tag{9}$$

and

$$\mathrm{PI}(k)\{\gtreqless\}PI(m) \text{ if } 2N(V_iV_j + V_iV_l + V_jV_l)\{\gtreqless\}N^3 + 4V_iV_jV_m, \tag{10}$$

A little arithmetic reveals some interesting properties. Suppose that voter m approves no proposition so that $V_m = 0$. Conditions (9) and (10) reduce to:

TABLE 1. Power Indices—5 Voters and 10 Propositions

Voter	Proclivity to Vote Yes	Power Index
i	.7	6.59
j	.6	6.89
k	.5	7.01
l	.4	6.89
m	.3	6.59

$$PI(k)\left\{\substack{> \\ = \\ <}\right\}PI(l) \text{ if } \frac{V_iV_j}{N^2}\left\{\substack{> \\ = \\ <}\right\}\tfrac{1}{2}, \tag{11}$$

and

$$PI(k)\left\{\substack{> \\ = \\ <}\right\}PI(m) \text{ if } \frac{V_iV_j}{N^2}\left\{\substack{> \\ = \\ <}\right\}\tfrac{1}{2} - \frac{V_l(V_i + V_j)}{N^2} + \frac{2V_iV_jV_l}{N^3}. \tag{12}$$

Assuming $V_i > V_j > V_k > V_l > V_m$, condition (11) says that the median voter's power index is greater than that of the relative negativist (voter l, as opposed to extreme negativist, voter m) if and only if the *product* of the relative and extreme affirmist's proclivities to affirm is greater than one-half. Consider the maximum value V_i can undertake, 10 (as there are only ten propositions). Then V_j must equal 6 or greater to give voter k, the median, a greater power index than l. This also implies via condition (12) that if $V_l = 0$ voter k dominates voter m. This result follows from our previous analysis of the three-voter case. If $V_i = 10$ and $V_m = 0$, then i and m cancel each other out leaving only three voters to determine the outcome.[9] In this case, voter j becomes the extreme affirmist and by our earlier result for three voters, the now-median voter dominates if V_j is greater than $N/2$, which is true when $V_j = 6$. Now suppose that $V_l = 1$ while $V_m = 0$. Condition (12) says if V_i assumes its maximum value, 10, V_j need only be equal to 5 for voter k to dominate voter m. Again, we can interpret this as saying voters i and m pair off. However, now that voter l is slightly more prone to vote yes, his affirming counterpart, voter j, need not be as affirmative in order for the median to have the highest power index. Hence, whether the median will have the highest power index depends on the *aggregate* "proclivity to vote yes" of the pairs of voters on either side of the median voter. For example, in the case where $V_m = 0$ and $V_l = 0$ any sum of expected yes votes equal to or greater than 16 will give voter k the most power. When $V_l = 1$, this sum drops to 15. Finally, as in the case of three voters, if all voters have a probability of voting yes (no) less than .5, then the absolute negativist (affirmist) dominates. These properties are illustrated in figure 2 where power indices for various distributions of voting proclivities are displayed. It should be obvious that if the

						i	*j*	*k*	*l*	*m*
Distribution of	10	4	2	1	0	1.24	4.89	8.59	8.78*	8.76
expected number	10	5	2	1	0	1.49	4.89	8.69*	8.69*	8.49
of yes votes	10	5	3	1	0	1.99	6.69	8.29*	8.29*	7.99
for voters *i, j,*	10	6	2	1	0	1.76	4.89	8.79*	8.59*	8.24
k, l, m	9	7	2	1	0	2.80	4.72	8.82*	8.56	8.16
	8	8	2	1	0	3.79	3.79	8.83*	8.55	8.14
	4	3	2	1	0	6.36	7.32	8.24	8.97	9.57*
	5	4	3	2	1	6.07	6.88	7.58	8.08	8.38*
	7	6	5	4	3	6.29	6.90	7.01*	6.90	6.29
	9	7	5	3	1	5.71	6.91	7.56*	6.91	5.71

Fig. 2. Power indices—5 voters and 10 propositions. Asterisk indicates highest power index for a given distribution.

orders of the distributions are reversed, the power indices will be in reverse order.

Generalized Information

Suppose that there is no information about individual proclivities to vote yes but there is general information about the whole group of voters. That is, only the proclivity of the *population* of voters to vote yes on a given proposition is known. Obviously, the notion of a particular voter with the median proclivity to vote yes vanishes. Some properties of the voting outcomes may, however, still be examined under these assumptions. For example, let there be a committee of seven voters casting 'yes' or 'no' votes on five independent proposals. Further, assume that the population proclivity to affirm is known, historically perhaps, to be .6 for the group. The probability of any proposal passing is the sum of the probabilities of getting exactly four yes votes, exactly five yes votes, exactly six yes votes, and exactly seven yes votes.[10] In the current example, this totals to .727. Notice the difference obtained here. The proportion of yes votes in the population over *all* issues in the set is .6. These votes, for lack of sufficient reason, can be thought of as randomly distributed across the five voters and the five issues. The probability that any *given issue* will obtain a majority exceeds the population probability that a *given vote* will be yes.

Table 2 shows probabilities of passage of a single proposition for alternative values of number of propositions to be voted on and number of voters when the general proclivity to vote yes is two-thirds. Fixing the number of propositions and the *proportion* of yes votes, an increase in the number of voters increases the probability of a proposal passing. On the other hand, increasing the number of propositions reduces (and apparently more slowly) the probability of any one proposition passing. Intuitively, as the number of voters becomes large given the number of propositions to be considered and

TABLE 2. Probability of a Given Proposition Passing for Various Numbers of Voters and Propositions and a Group Proclivity to Vote Yes of .67

Number of Propositions	Number of Voters			
	3	5	7	9
3	.773	.831	.871	.900
6	.755	.808	.847	.876
9	.748	.800	.838	.866
12	.747	.798	.836	.865

the group proclivity to affirm, the probability of any given proposal securing majority approval approaches 1(0) if the general proclivity to affirm is greater (less) than .5. This is because the random sample of votes drawn from the population approaches the population size and hence the expected number of yes votes in the sample will be close to the fraction of yes votes in the population. If this fraction is greater than one-half, a proposal will have majority approval.[11] On the other hand, as the number of propositions becomes large given the number of voters and the average proclivity to affirm, the probability of any given proposal passing approaches the average proclivity to affirm since one is drawing a sample of fixed size from an ever-increasing population.

An extension of this model is "casting lots" where there is a fixed number of yes votes in the group. There is no specific information on the number of yes votes held by each voter (cf. note 6). Assuming the yes votes are distributed randomly across voters and issues, the probability of any single issue passing remains the same as before. However, the probability of passage for subsequent proposals contingent on the passage of prior propositions falls as yes votes, or lots, are "used up" in passing the prior propositions. For example in our seven voter–five proposition case where there was a .6 group proclivity to affirm, let there be a total of 21 yes votes randomly distributed through the population of votes. In this case, the probability of any one proposal passing is .727 as before. The probability of any second proposal passing given one proposition has already passed is .687, and the probability of any third proposal passing given two have passed is .454. Table 3 lists the conditional probability of passage for all five propositions. The average probability for the set of five propositions is .482 which can be interpreted as the expected fraction of propositions under consideration that will pass given the voting scheme outlined above. Table 4 reports the expected fraction of propositions passed under the casting lots model for some selected voter-proposition combinations when 60 percent of the votes are yes votes. Table 5 reports expected fraction of propositions passed for the five voter–five proposition case given various percentages of yes votes in the total number of votes cast on all propositions.

TABLE 3. Probability of a Proposition Passing Conditional upon All Prior Propositions Have Passed—7 Voters and a Group Proclivity to Affirm of .6

Proposition	Probability of Passage
1st	.727
2d	.687
3d	.454
4th	.359
5th	.181

TABLE 4. Expected Fraction of Propositions Passed when There Is a .6 Group Proclivity to Affirm

Voters	Propositions
3	.355
5	.429
7	.482
9	.491

TABLE 5. Expected Fraction of Propositions Passed under Various Group Proclivities to Affirm Given 5 Voters and 5 Propositions

Number of Yes Votes in Total Votes Cast on All Propositions (in percentage)	Expected Fraction of Propositions Passed
10 (40%)	.091
15 (60%)	.429
20 (80%)	.922

As we saw earlier, the expected fraction of proposals passing, for a given percentage of yes votes in the population, rises with the number of voters and falls with the number of propositions. Not surprisingly, the expected passage rate increases with the percentage of yes votes in the population and appears to do so at an increasing rate.

Concluding Remarks

In conventional majority rule voting models, a weakening of the information concerning individual voter preference orderings effectively rules out any analysis of the voting outcomes. In the model of yes-no voting presented here, information assumptions can be progressively weakened without removing all of the model's predictive content.

We saw that when the rank ordering of the separate proposals are known and identical across voters, a unique outcome results and that this outcome exactly matches the median voter's preferred outcome. When ordinal rankings are assumed to differ among voters, majority rule voting still yields unique and stable results while the equivalence between the outcome and the median voter's preferences is weakened but not lost. That is, when the only information available is each voter's probability of voting yes, the voting outcome, under many circumstances, more closely corresponds to the preferences of the median voter than for any other voter. Finally, we considered the case where the only information is the voting group's aggregate likelihood to vote yes. Here, the median voter has no meaning and all that can be said is that the expected number of propositions passed by the group depends on the number considered, the number of voters, and the group proclivity to vote yes.

It should be pointed out that the results reported here are preliminary. Our arguments have been made by reference to specific examples rather than by formal theorems and proofs. One possible way of generalizing our model is to construct a large-scale computer simulation of the model under various information assumptions in order to discover the properties of the model when there are large numbers of voters and issues.[12]

Elsewhere (1981), we argued that the allocation of resources and the potential for economic growth will be affected differently when disputes over property are resolved privately or collectively by majority rule. When such disputes are resolved by yes-no voting such as deciding upon zoning variances, our analysis of yes-no voting outcomes becomes both relevant and important. It is interesting to speculate on the reasons for the relative analytical neglect of the voting institutions discussed here, which, as we have noted, are descriptive of frequently observed collective choice situations. Attention may have been concentrated on the single-election setting primarily because of the underlying motivation to answer questions concerning the 'rationality' or 'irrationality' of collective choice, questions that have been presumed to have normative or evaluative content. In straightforward yes-no majority rule voting on a proposition, there is a unique result. The question of "collective rationality" that has preoccupied public choice and social choice theorists simply does not arise. But the absence of this particular "rationality" question does not make the positive analysis of yes-no voting either uninteresting or unimportant, perhaps even for normative questions of institutional-constitutional design.

NOTES

We should like to thank our colleagues Geoffrey Brennan and Joseph Reid for helpful suggestions on earlier drafts of this paper, and Janet Faith, Richard Carter, and David

Laband for their research assistance. Finally, we thank Akira Yokoyama of Josai University for saving us from some mathematical errors.

1. We use the term *proposition* here rather than *issue* in order to avoid confusion with the treatment of multi-issue space in public choice analysis. In the latter, candidates, party platforms, and voters' ideal positions are described by vectors that employ several issue dimensions. But the analysis is directed toward the selection of a single candidate (a list), or platform, in a single election. Our analysis, by contrast, explicitly examines a bundle of "elections," or "group choices," each one of which is presented as a single yes-no proposition.
2. One might interpret this as a setting of an agenda by an exogenous actor. The voters themselves do not offer new proposals for consideration. Some of the recent work on agenda manipulation where voters are presented with only two options may readily be brought within our framework of analysis. See, for example, Romer and Rosenthal (1978), Mackay and Weaver (1978), and Denzau, Mackay, and Weaver (1981).

   An alternative interpretation is that the costs of communicating and enforcing compromise proposals are prohibitively high. This may particularly be the case when the issue is largely an emotional one such as the legalization of marijuana, the development of nuclear power, or the adoption of capital punishment. In other cases, the amount of wealth at stake simply may be insufficient to induce the affected party (or parties) to make expensive counterproposals.
3. This requires that propositions are defined such that "yes" has a consistent meaning. For example, a yes vote "to grant a zoning variance" is equivalent to a no vote "to not grant a zoning variance." We shall assume henceforth that propositions are worded so that affirmation is always denoted by a yes vote.
4. From both an empirical and practical viewpoint this question of correspondence of preferences with outcomes is interesting. Consider an individual board member. If every voting member had an established history or known frequency of voting yes or no on single issues drawn from a certain broadly defined set, then one might be able to predict the probable outcome of each vote. For a hypothetical board member, such information would be important particularly if he were new to the board and were trying to figure out on which proposals his vote would be most highly weighted in the outcome (cf. Badger 1972). By extension, a set of citizens searching for a person to represent them in some governmental body would want to know which of the candidates would have the most "power" to deliver legislation preferred by the constituency. In a rough sense, one might scan the voting proclivities of the current members of the legislative body and the proclivities of the candidates so as to decide upon the most effective candidate. The same may hold true within legislative bodies, like Congress, where newly elected representatives must be assigned to existing committees and subcommittees. In this case, the impact of one representative's purely random voting (a reasonable assumption if there is no prior information on the new representative) on the committee outcomes would be taken into account by the party leader in assigning legislators to committees.
5. If voter i votes yes on a proposition, then to secure majority approval it is required that either j or k or both vote yes. The probability that j will vote yes on any given proposition, that is for a randomly selected ordering of j's preferences, is .5. For voter k this same probability is .33. As there are 3 pairs of votes from j and k

which in conjunction with *i*'s yes vote will result in a majority—yes-yes, yes-no, no-yes—the probability of approval is $(\frac{1}{2}\cdot\frac{1}{3}) + (\frac{1}{2}\cdot\frac{2}{3}) + (\frac{1}{2}\cdot\frac{1}{3}) = .67$. To obtain a majority no, the necessary possibilities are yes-no, no-yes, and no-no; and the probability of majority disapproval given *i* votes 'no' is $(\frac{1}{2}\cdot\frac{2}{3}) + (\frac{1}{2}\cdot\frac{1}{3}) + (\frac{1}{2}\cdot\frac{2}{3}) = .83$.

6. An alternative voting model is to allocate each voter a fixed number of yes votes to cast over all propositions in the set. The probability of a subsequent proposal passing given a prior proposal has passed will then decrease since some yes votes are "used up" in passing the earlier one. Calculating probabilities in such an analytical setting is extremely time-consuming. For example, if there are five voters and six propositions, to calculate the probability of the sixth proposal passing given the first five have passed for just one voter requires over ten hours of computer execution time! Thus as a model with any empirical content along the lines suggested in note 4, this alternative formulation would be inappropriate. On the other hand, the probabilities discussed in the paper can be easily calculated.
7. This result is analogous to the "pairing off" condition for majority rule equilibrium with the median preference dominating. See Plott (1967). However, as expression (5) shows, in our model median dominance also requires that the extreme voters do not both have proclivities greater than one-half.
8. Since we are dealing with discrete changes in the parameters differential calculus is inappropriate for determining sensitivities. Thus, average rates of change are used where all changes in the independent variables are unit changes.
9. See note 7 above.
10. Letting L = total votes cast (voters times propositions), R = number of yes votes in L, D = number of voters, and k = number of yes votes in a random sample of D votes, the probability of getting exactly K yes votes on a given proposal is given by the hypergeometric probability:

$$k = \frac{\binom{D}{K}\binom{L-D}{R-k}}{\binom{L}{R}}$$

See Feller (1957, 42).

11. This result, in a somewhat different context, has been obtained by Kazmann (1973) and Grofman (1975).
12. In a sense our efforts are at a stage analogous to that of Tullock and Campbell (1970) who used simulation techniques to determine the probability of the occurrence of majority rule cycles as the number of voters and number of candidates change.

REFERENCES

Badger, W. W. "Political Individualism, Positional Preferences, and Optimal Decision-rules." In *Probability Models of Collective Choice*, edited by R. Niemi and H. Weisberg. Columbus: Merrill, 1972.

Buchanan, J. M., and Faith, R. L. "Entrepreneurship and Internalization of Externalities." *Journal of Law and Economics* 24 (April 1981): 95–111.

Denzau, A. T.; Mackay, R. J.; and Weaver, C. L. "On the Initiative-Referendum Option and the Control of Monopoly Government." In *Tax and Expenditure Limitations,* edited by Helen Ladd and T. N. Tideman, pp. 191–221. Washington, D.C.: Urban Institute Press, 1981.

Feller, W. *An Introduction to Probability Theory and Its Applications.* Vol. 1, 2d ed. New York: Wiley, 1957.

Grofman, B. "A Comment on 'Democratic Theory': A Preliminary Mathematical Model." *Public Choice* 21 (Spring 1975): 99–103.

Kazmann, R. G. "Democratic Organization: A Preliminary Mathematical Model." *Public Choice* 16 (Fall 1973): 17–26.

Mackay, R. J., and Weaver, C. L. "Monopoly Bureaus and Fiscal Outcomes: Deductive Models and Implications for Reform." In *Policy Analysis and Deductive Reasoning,* edited by G. Tullock and R. Wagner. Lexington, Mass.: D. C. Heath, 1978.

Plott, C. R. "A Notion of Equilibrium and Its Possibility under Majority Rule." *American Economic Review* 57 (September 1967): 787–806.

Romer, T., and Rosenthal, H. "Political Resource Allocation, Controlled Agendas, and the Status Quo." *Public Choice* 33 (Winter 1978): 27–43.

Tullock, G., and Campbell, C. D. "Computer Simulation of a Small Voting System." *Economic Journal* 80 (March 1970): 97–104.