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CHAPTER 1

Exploring Nonlinear Dynamics with a Spreadsheet: A Graphical View of Chaos for Beginners

L. Douglas Kiel and Euel Elliott

The mathematical foundation of chaos theory and the unique vernacular of this new science can deter some researchers from exploring the dynamics of nonlinear systems. Terms such as periodicity, sensitive dependence on initial conditions, and attractors are not the usual vernacular of the social sciences. However, the modern microcomputer and electronic spreadsheet software provide means for the novice to chaos research to explore the mathematics of chaos. The graphics capabilities of spreadsheet software also provide a visual means for exploring chaotic dynamics. This is particularly important considering the reliance of chaos researchers on graphical analysis. The intractability of nonlinear mathematics, while often defying solution, is now explored via visual analysis. This chapter should thus bring to light the amazing behavior and visual imagery of nonlinear dynamical systems and their relevance to social science.

Fortunately, the dynamics of time-based nonlinear systems can readily be explored by researchers new to the study of chaos theory. This exploration is accomplished here via the use of a simple algebraic formula and the computational and graphical powers of an electronic spreadsheet. The electronic spreadsheet readily allows the researcher to generate nonlinear time series and then examine these series graphically. Only a minimal knowledge of spreadsheets is necessary for the reader to follow the examples in this chapter. Readers are also urged to examine in greater detail the mathematical formulation and its various dynamics presented here.

By examining the chaotic and, more generally, nonlinear behavior in this chapter, the reader will understand that a simple deterministic equation can generate very complex behavior over time. This has considerable value for social scientists as we learn that systems evolve from the simple to the complex (Prigogine and Stengers 1984). This chapter also reveals the importance of history to social systems. The initial starting point of a social system has much to do with its eventual structure and behavior.

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As noted in the introduction to this volume, nonlinear systems can take on a wide array of behaviors over time. Scientists have, however, classified these behaviors into three distinct types of time-based regimes. These behavioral regimes are (1) convergence to an equilibrium or steady state; (2) periodic behavior or a stable oscillation; and (3) chaos. The most widely used mathematical formula for exploring these three behavioral regimes is a first-order nonlinear difference equation, labeled the logistic map. This mapping takes on the form

$$x_{t+1} = kx_t(1 - x_t)$$

The variable to be examined is the value x. The parameter, or boundary value, of the formula is a constant, k. Remember, chaotic behavior occurs within defined parameters. The subscript t represents time and is the current value of the variable x. The subscript $t + 1$ represents one time period of the variable x following the previous x_t.

Mapping this formula also requires an initial starting value. The starting point, usually called the initial condition, is represented by the first value of x_t, x_0. Once the first value of x_t and the parameter value are determined, a simple "copy" command with the spreadsheet can be used to generate the time series. The copy command serves the purpose of recursion or feedback by using the previous value to generate the current value of x_t.

A couple of rules must be followed when using the logistic map. First, the initial condition must be a fractional value such that $0 < x_0 < 1$. Second, the parameter value or constant, k, must be greater than 0 and less than 4. Adventurous readers are welcome to explore higher values of k. The following is the general framework for inputting the logistic map into a spreadsheet.

1. In cell A1, input a fractional value for x_0 between 0 and 1. This number is the initial condition.
2. In cell B1, input the k value or constant. Remember, this number must be greater than 0 and less than 4.
3. In cell A2, input the formula (\$B\$1·A1) · (1 − A1). Cell A2 represents the value x_{t+1}.
4. Next, copy cell A2 down to cell A61. This affords sixty iterations of the equation.
5. Then produce a line graph of the values in cells A1–A61 with the graph function in your spreadsheet. This graph provides a visual image of what happens as the system "evolves."
6. To change the dynamics of the time series, simply change the values in cells A1 (x) and B1 (k). To make a longer time series, just copy more cells down from A61.

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Stable Equilibrium

A fascinating aspect of the logistic map is that each behavioral regime occurs within defined mathematical boundaries. For example, values of k between 0 and 3 (Stewart 1989) will converge to an equilibrium. Table 1.1 reveals three recursions of the equation with the same initial condition (x_0) of 0.97, but with varying values of the parameter constant (k). This table reveals that the iterations rapidly converge to a steady state and do not leave this state, or mathematical point, once convergence occurs. Note that as the constant approaches three, convergence to a stability requires more iterations. Figure 1.1 reveals the graph generated for the constant (k) of 2.827 with 100 iterations.

TABLE 1.1. Numerical Iteration of Stable Equilibria from the Logistic Map

k = 1.95	k = 2.35	k = 2.65	k = 1.95	k = 2.35	k = 2.65
0.97	0.97	0.97	0.48717949	0.57446809	0.62264174
0.056745	0.068385	0.077115	0.48717949	0.57446809	0.62264136
0.10437376	0.14971496	0.18859593	0.48717949	0.57446809	0.62264161
0.18228576	0.29915591	0.40552289	0.48717949	0.57446809	0.62264145
0.29066244	0.49270488	0.6388463	0.48717949	0.57446809	0.62264155
0.40204669	0.58737494	0.61141252	0.48717949	0.57446809	0.62264148
0.46879004	0.56955921	0.62960622	0.48717949	0.57446809	0.62264153
0.48560058	0.57612956	0.61798591	0.48717949	0.57446809	0.6226415
0.48709568	0.57388008	0.62561021	0.48717949	0.57446809	0.62264152
0.48717528	0.57467307	0.6206885	0.48717949	0.57446809	0.6226415
0.48717928	0.57439624	0.62390086	0.48717949	0.57446809	0.62264151
0.48717948	0.57449322	0.62181873	0.48717949	0.57446809	0.62264151
0.48717949	0.57445929	0.62317452	0.48717949	0.57446809	0.62264151
0.48717949	0.57447116	0.6222943	0.48717949	0.57446809	0.62264151
0.48717949	0.57446701	0.62286688	0.48717949	0.57446809	0.62264151
0.48717949	0.57446846	0.62249489	0.48717949	0.57446809	0.62264151
0.48717949	0.57446795	0.62273676	0.48717949	0.57446809	0.62264151
0.48717949	0.57446813	0.62257957	0.48717949	0.57446809	0.62264151
0.48717949	0.57446807	0.62268176	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62261534	0.48717949	0.57446809	0.62264151
0.48717949	0.57446808	0.62265852	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62263045	0.48717949	0.57446809	0.62264151
0.48717949	0.57446808	0.62264869	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62263684	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62264455	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62263954	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62264279	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62264068	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62264205	0.48717949	0.57446809	0.62264151
0.48717949	0.57446809	0.62264116	0.48717949	0.57446809	0.62264151

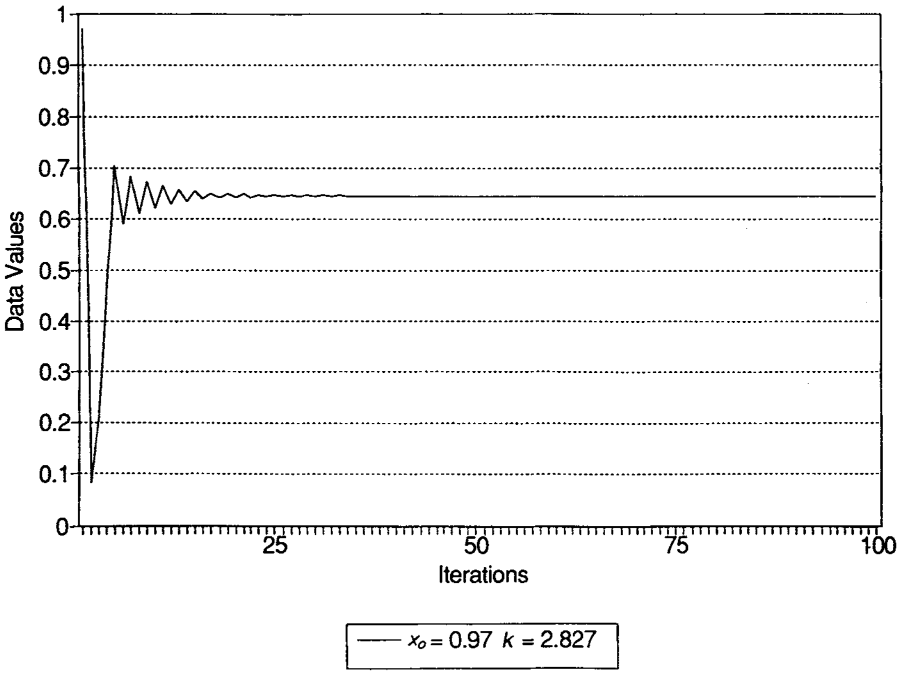

Fig. 1.1. Stable equilibrium

Periodic Behavior

A second type of nonlinear behavior that can occur over time is periodic behavior. Periodic behavior is cyclical or oscillatory behavior that repeats an identifiable pattern. Such periodic behavior starts to occur when $k > 3$. This regime initiates instability into the equation as the data start to oscillate. Such a change in the qualitative behavior of the time series is referred to as a bifurcation, or a branching to a new regime of behavior. This can be seen in column one of table 1.2. This first column represents a two-period cycle in which the value of x moves back and forth between two values. At approximately the (k) value of 3.5 (Stewart 1989) a four-period cycle occurs in which four numbers alternate in a consistent pattern. This four-period cycle is shown in column 2 of table 1.2 and is presented graphically in figure 1.2.

This process of cycles doubling in the number of alternating and continuous patterns of values is labeled period doubling. It is this continuous bifurcation of period doubling that eventuates in the "road to chaos" (Feigenbaum 1978). By exploring the range of (k) between 3.56 and 3.57, this period

TABLE 1.2. Numerical Iteration of Periodic Behavior from the Logistic Map

$k = 3.25$	$k = 3.5$	$k = 3.567$	$k = 3.25$	$k = 3.5$	$k = 3.567$
0.97	0.97	0.97	0.49526517	0.8269407	0.81057266
0.094575	0.10185	0.1037997	0.81242714	0.50088422	0.54769367
0.27829935	0.32016802	0.33182132	0.49526517	0.87499726	0.8836362
0.65275867	0.76181161	0.79086073	0.81242714	0.38281968	0.3667706
0.73666056	0.63509139	0.58998192	0.49526517	0.82694071	0.82843549
0.63047328	0.81112611	0.86286891	0.81242714	0.50088421	0.50697817
0.75717435	0.5362019	0.4220694	0.49526517	0.87499726	0.89157631
0.5975494	0.87041298	0.87008697	0.81242714	0.38281968	0.34481474
0.78157337	0.39477979	0.4031981	0.49526517	0.82694071	0.80584785
0.55482842	0.83625048	0.85832504	0.81242714	0.50088421	0.55808244
0.80273	0.47927466	0.43375848	0.49526517	0.87499726	0.87971647
0.51465229	0.87349661	0.87609822	0.81242714	0.38281968	0.37744353
0.81180226	0.38675099	0.3871983	0.49526517	0.82694071	0.83817334
0.49653289	0.83011131	0.8463627	0.81242714	0.50088421	0.48382357
0.81246093	0.49359283	0.46382729	0.49526517	0.87499726	0.8908166
0.49519654	0.87485632	0.88708271	0.81242714	0.38281968	0.34693493
0.81242501	0.38318959	0.35729561	0.49526517	0.82694071	0.80817906
0.49526949	0.82724365	0.81910968	0.81242714	0.50088421	0.55297655
0.81242727	0.50019058	0.52851887	0.49526517	0.87499726	0.88173916
0.4952649	0.87499987	0.88884887	0.81242714	0.38281968	0.37194968
0.81242713	0.38281283	0.35240733	0.49526517	0.82694071	0.83326232
0.49526519	0.82693509	0.81404791	0.81242714	0.50088421	0.49558553
0.81242714	0.50089707	0.53995074	0.49526517	0.87499726	0.89168049
0.49526517	0.87499718	0.88605685	0.81242714	0.38281968	0.34452367
0.81242714	0.38281989	0.36012471	0.49526517	0.82694071	0.80552531
0.49526517	0.82694088	0.8219613	0.81242714	0.50088421	0.55878584
0.81242714	0.50088382	0.52199806	0.49526517	0.87499726	0.87942325
0.49526517	0.87499727	0.89002388	0.81242714	0.38281968	0.37823754
0.81242714	0.38281968	0.34914287	0.49526517	0.82694071	0.83886531

doubling can be more closely examined. Table 1.2, column 3 shows the result of the value 3.567, an eight-period cycle (Stewart 1989). This process of period doubling continues, as k increases, to periods of 32, 64, 128, 256, and so on, until the onset of chaos.

Chaos

Chaotic behavior occurs at the (k) value of 3.8 to 4. This mathematical regime represents another clear bifurcation or qualitative change in a system's behavior. Three divergent values of (k) are shown in table 1.3 to show the diverse forms that chaotic regimes can take.

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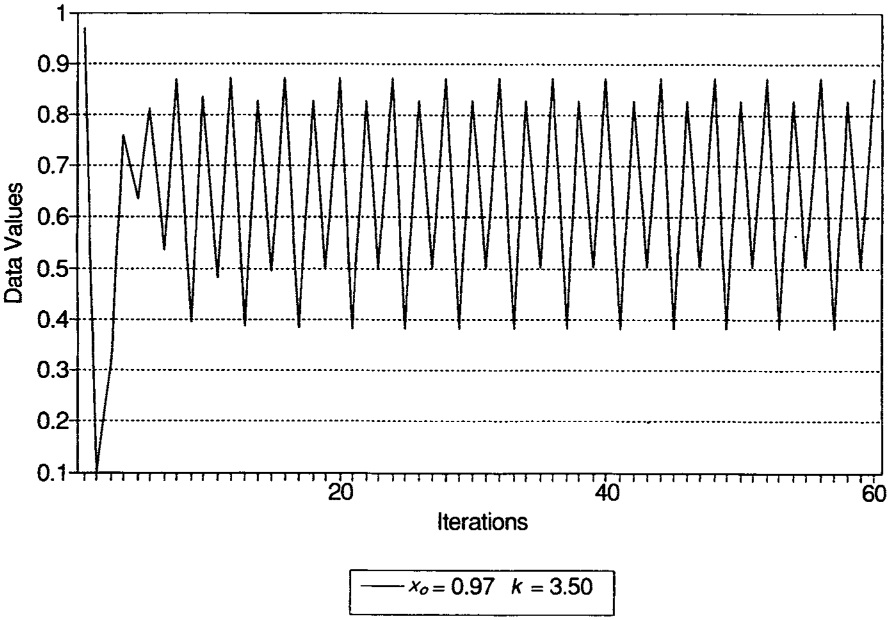

Fig. 1.2. Periodic behavior, four-period cycle

Figure 1.3 reveals the chaotic series when $k = 3.98$ and $x_0 = 0.90$. What distinguishes chaos from the other regimes of behavior is the lack of pattern in its longitudinal behavior. Chaotic behavior does not repeat itself and is thus labeled aperiodic. A close examination of the decimals in the values of (k) evidences this point. The reader will also note that chaotic behavior remains within definable parameters. While such chaotic behavior appears random, it is not. Chaos can be generated by a deterministic equation.

Sensitivity to Initial Conditions

In nonlinear dynamical systems, operating in a chaotic regime, small disturbances can have explosive and disproportionate (nonlinear) effects. While systems operating in a steady state or periodic regime will "damp" such disturbances, chaotic regimes tend to generate positive feedback and amplify such disturbances. This phenomenon of chaotic regimes is referred to as a sensitivity to initial conditions. In short, systems operating in chaotic regimes are very sensitive to small changes. It is this sensitivity that has generated the "butterfly metaphor" in chaos theory. Can the flapping of a butterfly's wings in

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TABLE 1.3. Numerical Iteration of Chaotic Behavior from the Logistic Map

k = 3.8	k = 3.89	k = 3.98	k = 3.8	k = 3.89	k = 3.98
0.9	0.9	0.9	0.18600293	0.88477839	0.18185933
0.342	0.3501	0.3582	0.57534218	0.39656834	0.59217033
0.8551368	0.88509166	0.91497318	0.92842951	0.93088436	0.96118843
0.47073584	0.39563016	0.30963308	0.25250299	0.25027741	0.14847484
0.94674571	0.93012599	0.85076653	0.71723187	0.72991427	0.50319163
0.19158941	0.25281746	0.5053121	0.77067919	0.76687238	0.99495946
0.58855506	0.73482408	0.99488769	0.67158454	0.69545082	0.01996024
0.92020041	0.75799626	0.02024297	0.83812323	0.82389802	0.07785607
0.27904015	0.71357355	0.07893611	0.51555619	0.56440038	0.28574213
0.76447163	0.79506285	0.2893667	0.94908042	0.95636658	0.81229239
0.68420808	0.63382848	0.81842177	0.18364175	0.16232792	0.60684438
0.82105605	0.90282986	0.59145814	0.56968635	0.52895274	0.94956543
0.55830744	0.34126233	0.96170892	0.93154649	0.96923916	0.19060589
0.93708092	0.87448115	0.14656298	0.24231699	0.11597882	0.61401563
0.22404903	0.42698145	0.49782745	0.69767797	0.39883289	0.94326173
0.66063403	0.95175965	0.99498121	0.801509	0.93268669	0.21300576
0.8519475	0.17860241	0.01987452	0.60455083	0.24422287	0.66718455
0.47930525	0.57067696	0.07752849	0.90846267	0.71800866	0.88375632
0.94837256	0.95306855	0.28464095	0.31600134	0.78761696	0.40886972
0.18605577	0.17399539	0.81040951	0.82134908	0.65070553	0.96194719
0.57546828	0.55907466	0.61151083	0.55759212	0.88414971	0.1456871
0.92835725	0.95892462	0.94551003	0.93739596	0.39844881	0.49536021
0.25273825	0.15322008	0.20505283	0.22300214	0.93238381	0.99491432
0.71767419	0.50470295	0.64876454	0.6584343	0.24524209	0.02013807
0.7699482	0.97241396	0.90691907	0.85461458	0.7200328	0.07853545
0.67308629	0.10434944	0.33597916	0.47214431	0.78416786	0.28802318
0.83615632	0.36356186	0.88792671	0.94705143	0.65837716	0.81616199
0.52059592	0.90008623	0.39606122	0.19055107	0.87492586	0.59716556
0.94838807	0.34983162	0.95200299	0.85611717	0.42568503	0.95742424

China create a tornado in Texas? In a nonlinear system such small occurrences may have massive results as behavior alters, changes, and perhaps, explodes over time. Furthermore, systems with very similar starting conditions in their evolutions may diverge to very different systems and structure over time. This point has obvious implications for social scientists as we explore how virtually identical systems generate unique histories.

The phenomenon of *sensitivity to initial conditions* can also be examined using the logistic map. This is best examined by comparing two time series with only slightly different initial conditions. By changing the initial condition by a mere one millionth (the last decimal place), a system with the same parameters or boundary values can show very different results over time. These two time series initially appear quite similar and actually map each

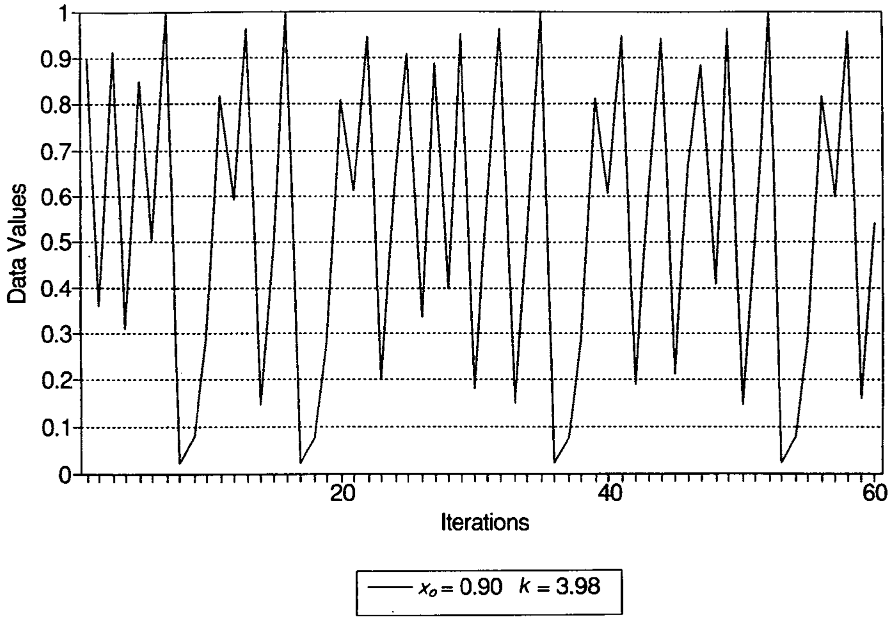

Fig. 1.3. Chaotic behavior

other perfectly. But once the divergence starts, the time series continue to behave quite differently, as can be seen in figure 1.4.

Attractors

Even researchers and students new to the study of chaos are familiar with the notion of *order in chaos*. The analyses of nonlinear time series show that a deeper underlying order exists in these diverse types of behavior. This order was discovered via graphical analysis of chaotic time series. Yes, to understand nonlinear systems, look at the pictures they generate.

This deeper order is discovered by an investigation of the *attractors* of a nonlinear system. Attractors provide a qualitative assessment of dynamic systems in motion (Mosekilde, Aracil, and Allen 1988, 21). Baumol and Benhabib (1989, 91) define an attractor as "a set of points toward which complicated time paths starting in its neighborhood are attracted." Pool (1989, 1292) defines an attractor as "the set of points in a phase space corresponding to all the different states of the system." More simply, the term attractor is used because the system's temporal evolution appears to be consis-

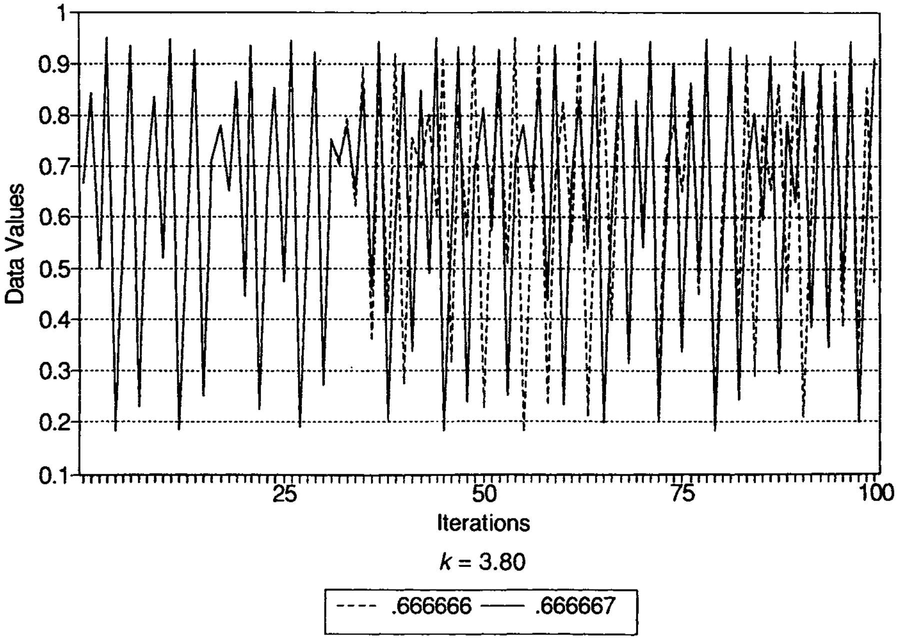

Fig. 1.4. Sensitivity to initial conditions

tently "pulled" to identifiable mathematical points. The attractor functions as an abstract representation of the flow, or motion, of a system. In short, the attractor stores information about a system's behavior over time. The attractor is used as a means for examining the structure of the underlying order within a nonlinear system.

The examination of an attractor is conducted by a mapping of the data onto a phase space (Thompson and Stewart 1986). A phase space represents a graphic backdrop for presenting the motion of time-based data. The examination of an attractor is conducted in a $t/t + 1$ phase space (Baumol and Benhabib 1989, 91). In this case, t (time) represents the current value of x, while $t + 1$ (time + 1) represents the next value of x. The t is plotted on the horizontal axis and $t + 1$ is plotted on the vertical axis. This method of plotting the data reveals the relationship between a previous period's measured result relative to the current time period's measured result. When plotting a one equation system such as the logistic map we refer to the phase space as a phase plane.

The attractors of nonlinear systems thus represent a dynamic structure

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that traces the longitudinal behavior of a system. Studies of these attractors, generated by various nonlinear equations, evidence an enormity of shapes and patterns (Gleick 1987). These attractors thus represent the structures that describe and dominate a nonlinear system during its evolution. Each of the three nonlinear regimes described above represent a unique attractor type. Steady state attractors converge to a point and remain there. Attractors from periodic regimes are called *limit cycles* and oscillate around a set of defined mathematical points, creating a circular pattern. A chaotic or *strange attractor* takes on a multitude of surprising shapes.

Attractors can be generated using an electronic spreadsheet. This is accomplished by generating an *XY* graph as follows:

1. Create a chaotic time series for about sixty iterations in column A. This series should be the horizontal axis (t).
2. In cell C1, place a zero. Then recreate the same chaotic series in column A, under the zero in column C. Make sure you put all values from column A, including the initial value, in column C. This series will be the vertical axis ($t + 1$).
3. Delete the last cell at the end of the column C ($t + 1$) to ensure each column has the same number of time periods.
4. Generate the graph to produce the attractor.

When the logistic map is operating in a chaotic regime it creates a hill-shaped attractor, or parabola. If the lines are deleted from the graph and only symbols are used, this underlying structure appears. This structure is the *order in chaos*. Given the nonlinearities that exist in social science data, social scientists may want to explore their data sets using phase space mappings. Researchers may find a wide range of attractors that describe the order in the apparent chaos of social science data.

Conclusion

The dynamism of social systems suggests that each behavioral regime noted above can appear within the long-term behavior of a nonlinear system. Because dynamical systems are historical systems they can reveal many types of behavior over time. Thus each behavioral type does not reflect permanent commitment only to that behavioral type, but rather reflects one possible type that may occur for a period during the life of a system.

Exploring the behavioral regimes of nonlinear systems should provide social scientists with a foundation for discovering such behavior in social phenomena. For example, notions of periodicity have always been used in analyses of society, as phenomena ranging from fads to vehicular traffic reveal such oscillatory behavior. Analysts may, however, have a more difficult time finding social systems that are truly in a steady state. Social systems do seem to oscillate at varying frequencies. Finally, chaos may provide a means for understanding the uncertain nature of social systems as both historical and evolving systems.