3

THEORY TO PRACTICE

In this chapter, we provide guidance for constructing statistical models that map onto substantive theory. We discuss the implementation of statistical analyses to test the theory, and we provide advice on interpreting empirical results.

Specifying Empirical Models to Reflect Interactive Hypotheses

Theory should guide empirical specification and analysis. Thus, for instance, empirical models of principal-agent and other shared-policy-control situations should reflect the convex-combinatorial form, with its multiple implied interactions, as described earlier. Theoretical models of behavior that suggest that institutional or environmental contexts shape the effect of individual characteristics on behaviors and attitudes should likewise specify empirical models that reflect the hypothesized context conditionality in interactions.

To facilitate discussion, we will provide empirical examples from a variety of substantive venues. Our first empirical example comes from Gary Cox's *Making Votes Count* (1997). (More examples from other substantive venues and illustrating interactions of other types of variables will appear later.) Cox's justifiably acclaimed book makes several institutional arguments in which some political outcome, y, say, the effective number of parties elected to a legislature or the effective number of presidential candidates, is a function of some structural condition, x, say, the number of societal groups created by the pattern of social cleavages (e.g., the effective number of ethnic groups), and some institutional condition, z, say, the proportionality or district magnitude of the electoral system or the presence or absence of a presidential runoff system. Theory in this case very clearly implies that the relationship between y and x should be conditional upon z and, conversely, that the relationship between y and z should be conditional upon x. As Cox (1997) theorizes, for example, "A polity will have many parties only if it *both* has many cleavages *and* has a permissive enough electoral system to allow political entrepreneurs to base separate parties on these cleavages. Or, to turn the formulation around, a polity can have few parties either because it has no need for many (few cleavages) or poor opportunities to create many (a constraining electoral system)" (206). (See, Amorim Neto and Cox 1997; Cox 1997; Ordeshook and Shvetsova 1994 for empirical implementation.)

The standard linear-interactive model can reflect the propositions that x and z affect y and that the effects of x and of z on y each depend on the other variable. One simple way to write this (compound) proposition into a linear-regression model is to begin with a standard linear-additive model expressing a relation from x and z to y, along with an intercept, and then to allow the intercept and the coefficients on x and z each to depend on the level of z and x:[1]

$$\begin{aligned} y &= \beta_0 + \beta_1 x + \beta_2 z + \varepsilon \qquad (1)\\ \beta_0 &= \gamma_0 + \gamma_1 x + \gamma_2 z \\ \beta_1 &= \delta_1 + \delta_2 z \\ \beta_2 &= \delta_3 + \delta_4 x \end{aligned}$$

Combining these equations, we may express the model of y for estimation by linear regression in the standard linear-interactive manner:

$$y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz} xz + \varepsilon \qquad (2)$$

As originally expressed in (1), the coefficients in this linear-interactive model (2) are $\beta_x = \gamma_1 + \delta_1$, $\beta_z = \gamma_2 + \delta_3$, $\beta_{xz} = \delta_2 + \delta_4$. More important, in this model, the effects of x and z on y depend on z and x, respectively, as an interactive theory would suggest.

Theory or substance might suggest several alternative routes to this same general model. For example, suppose we were to specify a system

1. We begin with the simplest case, where the effects of x and of z are deterministically dependent on, respectively, z and x. Subsequently, we relax this assumption to discuss probabilistic dependence (i.e., with error).

of relationships in which the effect of x on y and the intercept (conditional mean of y) depend on z:

$$\begin{aligned} y &= \beta_0 + \beta_1 x + \varepsilon \\ \beta_0 &= \gamma_0 + \gamma_1 z \\ \beta_1 &= \delta_1 + \delta_2 z \end{aligned} \tag{3}$$

This is a common starting point for "multilevel" models in which some individual (microlevel) characteristic, x, is thought to produce microlevel outcomes or behaviors, y, although the mean of that outcome or behavior, β_0, and the effect, β_1, of that individual characteristic, x, may vary across contexts, which are described by a macrolevel characteristic, z. Combining these equations, we may express the following model for y:

$$y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz} xz + \varepsilon \tag{4}$$

where $\beta_x = \delta_1$, $\beta_z = \gamma_1$, $\beta_{xz} = \delta_2$.

Note that the models actually estimated in (2) and (4) are identical, even though the theoretical/substantive stories told to derive the models from (1) and (3) seem to differ.[2] Each of these seemingly different theoretical stories yields the same mathematical model: the linear-interactive model (2).[3] This result demonstrates that, although the substance may determine which of these arguments is most intuitive to express, the distinction cannot be drawn mathematically. This mathematical ambiguity arises because the propositions being modeled are logically symmetric; that is, these statements all logically imply each other, and, in that sense, *they are identical; they cannot be distinguished because they are not distinct.* As Fisher (1988) writes, "prior theoretical specification is needed to interpret [in this sense] regression equations with product terms" (106). We concur but stress that the interpretive issues here are presentational and semantic because the alternatives are logical equivalents. These alternative theoretical stories may sound different in some substantive contexts, and some versions may seem more intuitive to grasp in certain contexts and others in other contexts. However, they are not actually alternatives; they are all the same tale.

2. To complete the list: a model in which y is a linear-additive function of z and the effect of z and the intercept depends on x, or one where the effect of x depends on z or the effect of z depends on x (and each effect may be nonzero when the other variable equals zero), also produces this same linear-interactive regression model.

3. Note: the linear-interactive model is not the only model form that would imply that the effects of x depend on z and vice versa, but, absent further theoretical elaboration that might suggest a more specific form of interaction, additive linear-interactive models like (2) are the logical, simple default in the literature.

Alternatively, one could propose a substantive argument that the effect of x on y depends on z but that z matters for y only insofar as it alters the impact of x and, in particular, z has no effect when x is equal to zero (not present). This *is* a change in the theoretical account of the relationship between the variables; it is a logically distinct argument, and it produces a truly different equation to be estimated:

$$\begin{aligned} y &= \beta_0 + \beta_1 x + \varepsilon \\ \beta_1 &= \delta_1 + \delta_2 z \\ y &= \beta_0 + \beta_x x + \beta_{xz} xz + \varepsilon \end{aligned} \tag{5}$$

where $\beta_x = \delta_1$, $\beta_{xz} = \delta_2$.

In this system of equations, we again see that z conditions the effect that x has on y and vice versa. However, by theoretical claim and ensuing model construction, z's sole effect is to determine the effect of x on y, and, in particular, movements in z have no effect on y when x is zero.[4] Scholars will typically think of z in this scenario as the *intervening variable:* intervening in x's relationship to y. However, notice that just as a value of x exists, namely, $x = 0$, where the effect of z is zero, a value of z exists, namely, $z = -\beta_x/\beta_{xz}$, where the effect of x is zero. The substance of the context at hand may suggest whether to conceive $x = 0$ or $z = -\beta_x/\beta_{xz}$, or, for that matter, some other value of x or z, as the base from which to decide whether x or z is the one *intervening* in the other's relationship with y. Mathematically that determination is once again arbitrary because, *logically, all interactions are symmetric.*[5] Given this logical symmetry, x and z must necessarily *both* intervene in the other's relationship to y. In this sense, the language of one variable being the intervening or moderating variable and the other being the one moderated may be best avoided; if an interaction exists, then all variables involved intervene or moderate in the others' relations to y.

The preceding equations assume that the effect of x on y depends on z and the effect of z on y depends on x *deterministically,* that is, without error. This might seem odd, given that our model proposes that x and z predict y only with error (hence the inclusion of the term ε), but the subsequent equations propose that the effect of x on y and of z on y each

4. We discuss this type of model further in the first section of chapter 4.

5. Mathematically, the proof of this logically necessary symmetry in all interactions is simply

$$\frac{\partial\left(\frac{\partial f(x,z)}{\partial x}\right)}{\partial z} \equiv \frac{\partial^2 f(x,z)}{\partial x \partial z} \equiv \frac{\partial^2 f(x,z)}{\partial z \partial x} \equiv \frac{\partial\left(\frac{\partial f(x,z)}{\partial z}\right)}{\partial x} \quad \forall\, f(x,z).$$

depend on the other variable *without error.* We can easily amend the linear-interactive model to allow a more logically consistent stochastic conditioning of the effects of variables by the others' levels thus:

$$
\begin{aligned}
y &= \beta_0 + \beta_1 x + \beta_2 z + \varepsilon \\
\beta_0 &= \gamma_0 + \gamma_1 x + \gamma_2 z + \varepsilon_0 \\
\beta_1 &= \delta_1 + \delta_2 z + \varepsilon_1 \\
\beta_2 &= \delta_3 + \delta_4 x + \varepsilon_2
\end{aligned}
$$

Combining these equations allows expressing y for regression analysis in the now-familiar

$$y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz} xz + \varepsilon^* \tag{6}$$

where $\varepsilon^* = \varepsilon + \varepsilon_0 + \varepsilon_1 x + \varepsilon_2 z$, $\beta_x = \gamma_1 + \delta_1$, $\beta_z = \gamma_2 + \delta_3$, $\beta_{xz} = \delta_2 + \delta_4$.[6]

The composite residual ε^*in (6) retains zero expected value and noncovariance with the regressors x, z, and xz provided that its components, ε, ε_0, ε_1, and ε_2, do so. These key assumptions of the classical linear-regression model (CLRM) ensure unbiasedness and consistency of ordinary least squares (OLS) coefficient estimates. However, this compound residual does not retain constant variance, since it waxes and wanes as a function of x and z. This heteroskedasticity undermines the efficiency of the OLS coefficient estimates and the accuracy of OLS standard errors. In other words, if the conditionalities of the x and z relationships with y themselves contain error, then the standard linear-interactive model has heteroskedastic error even if the individual stochastic terms comprising its compound residual are homoskedastic. Thus, OLS coefficient estimates are unbiased and consistent but not efficient. The OLS standard-error estimates, on the other hand, are incorrect,[7] but, as we show later, these problems are often easy to redress satisfactorily with familiar techniques. We return to this technical concern in the section "Random-Effects Models and Hierarchical Models" in chapter 5, because this concern often underlies calls for random-coefficient or linear-hierarchical

6. Note that the terms involving $\varepsilon_1 x$ and $\varepsilon_2 z$ can be removed from the expression for the composite error, ε^*, and replaced by appending $+\varepsilon_1$ to the expression for β_x and $+\varepsilon_2$ to that for β_z, to give another common expression of the random-coefficients/random-effects model.

7. To be precise, OLS standard-error estimates, as estimates of the true variation across repeated samples of the OLS coefficient estimates under the CLRM assumptions, are always inefficient in the presence of any heteroskedasticity, and, when the heteroskedasticity is a function of the regressors, as is the case here, they are biased and inconsistent as well.

models. For now, we proceed assuming the researcher estimates a model like (4) by OLS.

Let us return to our example of electoral systems, social cleavages, and the number of parties or candidates to illustrate the preceding discussion. We follow Cox's analysis of the effects of presidential-runoff systems (*Runoff*) and the effective number of ethnic groups in a society (*Groups*) on the effective number of presidential candidates (*Candidates*) that emerges in a presidential democracy.[8] The theory suggests that the impact of social cleavages on the effective number of candidates depends on whether a runoff system is used and, symmetrically, that the impact of the runoff system on the effective number of candidates depends on the number of societal groups. (Recall that these are logically two sides of the same proposition.) The confluence of a high number of social cleavages and a runoff system is hypothesized to produce a high effective number of presidential candidates, because the number of societal groups increases the potential number of parties and the runoff system attenuates the incentives for preelection coalition building between such groups. Given this theoretical structure, we can specify the following model for estimation:

$$Candidates = \beta_0 + \beta_G Groups + \beta_R Runoff + \beta_{GR} Groups \times Runoff + \varepsilon \tag{7}$$

The data set includes information from sixteen presidential democracies in 1985.[9] The dependent variable, *Candidates*, the effective number of presidential candidates, ranges from 1.958 to 5.689, with a mean of 3.156 and a standard deviation of 1.202. The independent variable, *Groups*, the effective number of ethnic groups in a society,[10] ranges from 1 to 2.756, with a mean of 1.578 and a standard deviation of 0.630. The independent variable, *Runoff*, indicates the presence or absence of a runoff system for the presidential election; this dummy variable takes the value of zero if the system does not employ runoffs and one if it does use

8. Effective numbers are simply size-weighted counts of items. The effective number of social groups, for example, is $\left(\sum_{i=1}^{n} g_i^2\right)^{-1}$, where g_i is the group i's fraction of the population. The effective number of candidates is $\left(\sum_{i=1}^{n} v_i^2\right)^{-1}$, where v_i is candidate i's fraction of the vote total.

9. We selected this data set because it is freely available (at http://dodgson.ucsd.edu/lij/pubs/) so researchers can easily replicate our results and because of its very manageable size. The small *N*, however, makes finding any strong statistical significance rather unlikely, but weak significance hardly hampers our pedagogical purposes.

10. To avoid some tiresome repetition, we henceforth drop the adjectives *effective,* although they remain applicable.

them. In this sample of sixteen presidential democracies, exactly half have a runoff system. The OLS regression results appear in table 1.

How do we interpret these results? What do these estimated coefficients mean? The next section provides guidance on these questions.

Interpreting Coefficients from Interactive Models

In the simple linear-additive regression, $y = \beta_0 + \beta_x x + \beta_z z + \varepsilon$, the effect of the variable, x, on y is simply its coefficient, β_x. When x rises by one unit, ceteris paribus, y rises by β_x. Likewise for z, its effect on y is its coefficient, β_z. In this case—and *only* in the purely linear-additive regression case—the *coefficient* on a variable and the *effect* on the dependent variable of a unit increase in that independent variable (ceteris paribus and controlling for other regressors) are identical.

In interactive models, as in all models beyond the strictly linear-additive, this equivalence of coefficient and effect no longer holds. In an attempt to cope with this change, current practice in interpreting interactive effects often substitutes some vague and potentially misleading terms, such as *main effects* and *interactive effect, direct effects* and *indirect effect,* and *independent effects* and *total effect,* for the coefficients on

TABLE 1. OLS Regression Results, *Number of Presidential Candidates*

	Coefficient (standard error) p-Value
Ethnic Groups	−0.979
	(0.770)
	0.228
Runoff	−2.491
	(1.561)
	0.136
Ethnic Groups × Runoff	2.005
	(0.941)
	0.054
Intercept	4.303
	(1.229)
	0.004
N (degrees of freedom)	16 (12)
Adjusted R^2	0.203
$P > F$	0.132

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided p-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

x and z in the first case and on xz in the second. Such terminology is usually unhelpful at best, misleading or incorrect at worst.[11]

Instead, we encourage researchers to recall that each variable involved in the interaction terms of interactive models has multiple effects, not any single, constant effect, such as might be given somehow by a single coefficient, nor a *main* effect and an *interactive* effect, such as might be given by some pair of coefficients, but multiple, different effects depending on the levels of the other variable(s) with which it interacts. When a researcher argues that the effect of some variable x on y depends on z, he or she is arguing that x has different effects on y, depending on the specific values of z. In the interactive case, the effects of x on y are therefore not any single constant, like the coefficient β_x on x in the simple linear-additive model. The effects of x on y vary. They depend on the coefficients on x and xz, as well as the value of z. To restate the general principle: outside of the purely linear-additive model, *coefficients are not effects*. The effect of x on y, as we elaborate subsequently, is the derivative, $\partial y/\partial x$, or the difference/change, $\Delta y/\Delta x$, which will only equal the coefficient on x by itself in the purely linear-additive case.

Terming one coefficient the *main* effect and another the *interactive* effect thus perilously confuses coefficients for effects. Substantively, there may in fact be nothing whatsoever "main" or "direct" about the particular effect to which the coefficient on x actually does refer. Researchers cannot appropriately refer to the coefficient on x as "the main effect of x" or "the effect of x . . . independent of z" or "considered independently of z" or, certainly not, "controlling for z." The coefficient on x is just one effect x may have, namely, the effect of x at $z = 0$. That is, the coefficient on x gives the estimated effect of a unit change in x, *holding* z *fixed at zero*. We note that this zero value of z may have nothing at all "main" about it. It may fall outside the range of what appears in the sample, or it could even be logically impossible! The effect of x on y at $z = 0$ is obviously not "independent of z"; in fact, it is connected with a particular value of z. This effect of x on y when $z = 0$ is also a very different thing from the effect of x on y "controlling for z." The simple linear-additive multiple-regression model estimates a single, constant "effect of x on y, controlling for z." The linear-interactive model estimates the effect of x on y as a function of z.

Our empirical example illustrates and clarifies these points. The esti-

11. Note that some of this terminology also refers to path-analytic models, which specify that some variable x affects the level (rather than, or in addition to, the effect) of some variable z that then determines y. This overlap in terminology provides even more confusion for the researcher.

mated coefficient on *Runoff* ($\hat{\beta}_R = -2.491$) gives the estimated effect of runoff elections on the number of presidential candidates for the specific case where *Groups* takes a value of zero. But the number of societal groups never takes the value of zero in the sample; in fact, the number of ethnic groups in a society cannot logically equal zero. Thus, an interpretation of $\hat{\beta}_R$, the estimated coefficient on *Runoff*, as the "main" effect of a runoff system is nonsensical; far from a "main" effect, this is actually the effect at a value of ethnic heterogeneity that does not, and indeed could not, exist.

If, however, *Groups* were rescaled to include a value of zero, for example, by subtracting some constant value, such as the mean, and calling the resulting variable *Groups**, then the estimated coefficient $\hat{\beta}_{R^*}$ would be the estimated effect of *Runoff* when the rescaled variable *Groups** takes the value of zero. This is assuredly logically possible and in sample now, but the notion that the effect at this particular substantive value of ethnic heterogeneity is somehow "main" would remain strained and potentially misleading. That the effect of some variable when its moderating variable happens to be at its mean should be called a "main effect" while all the other effects at all the other logically permissible or empirically existent values are something other than "main" seems an unnecessary and possibly misleading substantive imposition, especially since the theoretical and substantive point of the interaction model in the first place is that the effects of the interacting variables vary depending on each other's values. We return to this topic of mean-rescaling interactive variables in the first section of chapter 4.

Symmetrically, the estimated coefficient $\hat{\beta}_G$, the coefficient on *Groups*, refers to our estimate of the effect of the number of ethnic groups when *Runoff* equals zero. This value does logically and empirically exist, and so the estimated value of $\hat{\beta}_G = -0.979$ tells us something substantively relevant. It reports an estimate that, in a system without runoffs, the number of ethnic groups has a negative impact on the number of presidential candidates. Specifically, an increase of 1 in the number of ethnic groups is empirically associated with a 0.979 reduction in the number of presidential candidates, *in systems without runoff elections*. (We find this result substantively puzzling, but that is the estimate.) Note, however, that the coefficient $\hat{\beta}_G$ only tells part of the story—it only reveals the estimated effect of *Groups* in one condition: when *Runoff* equals zero.

The researcher who equates a coefficient in an interactive model to an effect is thus treading on hazardous ground. At best, the researcher will be telling a story about an effect that applies to only one of several possible conditions (e.g., when $z = 0$ or when $z = \bar{z}$). At worst, the researcher will be telling a story about an effect that applies in no logically possible condition—an effect that is logically meaningless. In short, put simply, and reiterating this crucial point: outside the simplest purely linear-additive case, *coefficients* and *effects* are different things.

We suggest two effective and appropriate methods of interpreting results from interactive models: differentiation (which requires working knowledge of entry-level calculus) and differences in predicted values (which does not).

Interpreting Effects through Differentiation

Consider the following standard linear-interactive regression model:

$$y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz} xz + \varepsilon \tag{8}$$

The effects of an independent variable, x, on the dependent variable, y, can be calculated by taking the first derivative of y with respect to x (as suggested by, e.g., Friedrich 1982; Stolzenberg 1979). This is a direct and simple means of identifying the effects of x on y or the effects of z on y because first derivatives or first differences, $\partial y/\partial x$ and $\partial y/\partial z$, or $\Delta y/\Delta x$ and $\Delta y/\Delta z$, *are* effects. One may, in fact, read $\partial y/\partial x$ (or $\Delta y/\Delta x$), for example, as "the change in y, ∂y (or Δy), induced by a marginal (derivative) or unit (difference) increase in x, ∂x (or Δx), all else held constant." Differentiation is a simple, reliable, methodical way of calculating interactive effects. To help it fulfill its promise of simplicity and to reduce the tendency to induce mistakes, we provide a table of basic differentiation rules in appendix A.

In the standard linear-interactive model (8), the first derivatives of y with respect to x and z are

$$\partial y/\partial x = \beta_x + \beta_{xz} z \tag{9}$$

$$\partial y/\partial z = \beta_z + \beta_{xz} x \tag{10}$$

As (9) and (10) exemplify, the first derivative of (8) with respect to x and z yields the conditional effect of those variables directly. Derivatives *are* effects, whether in basic linear-additive regression models, when they yield just the coefficient on the variable of interest, or in linear-interactive models like (8), when they give expressions like (9) and (10) involving two coefficients and the other interacting variable. This generalizes to any other model regardless of its functional form.

The effect of x on y in an interactive model like (8) is $\beta_x + \beta_{xz}z$, which reflects the conditional argument underlying that model. As noted earlier, β_x is merely the effect of x on y when z happens to equal zero, and it is neither necessarily the "main" effect in any sense nor the effect "independent of" or "controlling for" z. Nor, we now add, does β_{xz} embody the "interactive" effect of x or of z exactly, as often suggested. The coefficient β_{xz} indicates by how much the effect of x on y changes per unit increase in z. It also indicates the logically and mathematically identical amount by which a unit increase in x changes the effect of z on y. Neither is precisely an effect. They are statements of how an effect *changes:* that is, an effect on an effect. The sign and magnitude of β_{xz} tell us how the effect of x on y varies according to values of z. In an interactive model, indeed in any model, the effect of a variable, x, on y is $\partial y/\partial x$. Here that effect is $\beta_x + \beta_{xz}z$. One cannot distinguish some part of this conditional effect as main and another part as interactive.

Returning to our empirical example of the interaction between institutional structure and social cleavages in determining the number of presidential candidates, we are now prepared to interpret the results using differentiation. Recall the results from our OLS regression:[12]

$$\widehat{Candidates} = 4.303 - 0.979(Groups) - 2.491(Runoff) + 2.005(Groups \times Runoff) \tag{11}$$

Applying (9) and (10), we see that

$$\partial\hat{y}/\partial G = -0.979 + 2.005(Runoff) \tag{12}$$

$$\partial\hat{y}/\partial R = -2.491 + 2.005(Groups) \tag{13}$$

Thus, the effect of societal groups on the number of presidential candidates varies with the presence or absence of a runoff, and the effect of a runoff on the number of presidential candidates varies with the number of ethnic groups in society. These conditional effects can be easily calculated by inserting substantively relevant values for the variables of interest into equations (12) and (13).

Recall that *Runoff* takes only two values: zero in the absence and one in the presence of a runoff system. Hence, we use (12) to recalculate the

12. Although, technically, one cannot strictly differentiate with respect to noncontinuous variables, such as dummy variables, one can proceed ignoring this technicality without being misled. (Do remember, however, that *marginal* increases cannot actually occur, only *unit* increases from zero to one can.) Alternatively, one can calculate differences in predicted values, which we discuss next. For more detail, see note 17.

conditional effect of *Groups* on the number of candidates for these two substantively relevant values of *Runoff*. When *Runoff* = 0, $\partial\hat{y}/\partial G = -0.979 + 2.005 \times 0 = -0.979$. When *Runoff* = 1, $\partial\hat{y}/\partial G = -0.979 + 2.005 \times 1 = 1.026$. In the absence of a runoff, the estimated effect of ethnic groups is negative; in the presence of a runoff, the estimated effect of ethnic groups is positive. (The "Linking Statistical Tests with Interactive Hypotheses" section of this chapter discusses the standard errors and statistical significance of these estimated effects, which, like the effects themselves, vary with the level of the conditioning variable.)

Symmetrically, we can calculate the conditional effect of *Runoff* on the number of presidential candidates by inserting substantively relevant values of *Groups* into (13). Recall that *Groups* ranges from 1 to 2.756 in our data set. We should present the estimated effects of *Runoff* over a substantively revealing set of values for *Groups*: for example, over the sample range of values of *Groups*; or at evenly spaced intervals starting from the sample minimum to some substantively meaningful and revealing high value; or at the minimum, mean, and maximum; or at the mean, the mean plus and the mean minus a standard deviation or two.

To take one of these options, we calculate conditional effects when *Groups* ranges from 1 to 3, at evenly spaced intervals of 0.5, which yields the following estimated conditional effects:[13]

$$\text{When Groups} = 1: \quad \partial\hat{y}/\partial R = -2.491 + 2.005 \times 1 = -0.486$$

$$\text{When Groups} = 1.5: \quad \partial\hat{y}/\partial R = -2.491 + 2.005 \times 1.5 = 0.517$$

$$\text{When Groups} = 2: \quad \partial\hat{y}/\partial R = -2.491 + 2.005 \times 2 = 1.520$$

$$\text{When Groups} = 2.5: \quad \partial\hat{y}/\partial R = -2.491 + 2.005 \times 2.5 = 2.522$$

$$\text{When Groups} = 3: \quad \partial\hat{y}/\partial R = -2.491 + 2.005 \times 3 = 3.525$$

At the sample minimum (when the society has only one ethnic group), a runoff system has a negative effect on the number of presidential candidates (which, again, seems substantively odd), but as the number of ethnic groups rises, the runoff begins to affect the number of presidential candidates positively (which is more sensible). The size of the effect grows as ethnic groups become more numerous (also sensible). Again, the standard errors of these estimated effects and whether the effects are statistically significant are matters we will discuss subsequently.

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13. Although the sample maximum is 2.756, *Ethnic Groups* does extend beyond this value in some of the nonpresidential systems that Cox (1997) analyzes.

Interpreting Effects through Differences in Predicted Values

A second strategy for examining the effects of x and z on y consists of examining differences in predicted values of y for logically relevant and substantively meaningful values of x and z. This strategy does not require the researcher to have any knowledge of calculus; it is a bit more tedious but quite serviceable in its own right. Predicted values of y, denoted as $\hat{y}$, can be calculated by substituting the estimated values for the coefficients along with logically relevant and substantively revealing values of the covariates of interest into the theoretical model (equation (8)) and substituting in estimated coefficient values:

$$\hat{y} = \hat{\gamma}_0 + \hat{\beta}_x x + \hat{\beta}_z z + \hat{\beta}_{xz} xz \tag{14}$$

We can now calculate $\hat{y}$ at varying values of x (between, say, x_a and x_c) while holding z constant at some meaningful value (e.g., its mean value or some other substantively relevant value; if z is a dummy, for example, zero and one are meaningful). By doing so, the researcher can calculate how changes in x (from x_a to x_c) cause changes in $\hat{y}$ (from $\hat{y}_a$ to $\hat{y}_c$). Recall that as x changes from x_a to x_c, while z is held at some meaningful value, say, z_0, this also implies that xz changes from $x_a z_0$ to $x_c z_0$. The predicted values, $\hat{y}_a$ and $\hat{y}_c$, can be calculated as follows:

$$\hat{y}_a = \hat{\gamma}_0 + \hat{\beta}_x x_a + \hat{\beta}_z z_0 + \hat{\beta}_{xz} x_a z_0 \quad \text{and}$$
$$\hat{y}_c = \hat{\gamma}_0 + \hat{\beta}_x x_c + \hat{\beta}_z z_0 + \hat{\beta}_{xz} x_c z_0$$

The change in predicted values can be calculated as the difference between $\hat{y}_a$ and $\hat{y}_c$:

$$\hat{y}_c - \hat{y}_a = \hat{\gamma}_0 + \hat{\beta}_x x_c + \hat{\beta}_z z_0 + \hat{\beta}_{xz} x_c z_0 - (\hat{\gamma}_0 + \hat{\beta}_x x_a + \hat{\beta}_z z_0 + \hat{\beta}_{xz} x_a z_0)\ \hat{y}_c - \hat{y}_a = \hat{\beta}_x (x_c - x_a) + \hat{\beta}_{xz} z_0 (x_c - x_a) \tag{15}$$

Symmetrically, the researcher can identify how $\hat{y}$ moves with changes in z (and xz) when x is held at some meaningful value.

In our example, we can examine how the predicted number of presidential candidates changes as we increase the number of ethnic groups in the presence and in the absence of runoff elections:

$$\widehat{Candidates} = 4.303 - 0.979(Groups) - 2.491(Runoff) + 2.005(Groups \times Runoff)$$

When $Groups = 1$ and $Runoff = 0$, we calculate the predicted number of candidates, $\hat{y}$ as

$$(\hat{y} \mid Groups = 1, Runoff = 0) = 4.303 - 0.979 \times 1 - 2.491 \times 0 + 2.005 \times 1 \times 0 = 3.324$$

TABLE 2. Predicted *Number of Presidential Candidates*

	Runoff = 0	*Runoff* = 1
Groups = 1	3.324	2.838
Groups = 1.5	2.835	3.351
Groups = 2	2.345	3.865
Groups = 2.5	1.855	4.378
Groups = 3	1.366	4.891

Table 2 presents the predicted number of candidates, as *Groups* ranges from one to three, when *Runoff* takes values of zero and one.

From such a table, the researcher can discern how the independent variables, *Groups* and *Runoff*, influence the predicted dependent variable, *Candidates*. By looking across single rows, we see the effect of the presence of a runoff system at the given number of *Groups* associated with each row. In the first row, for example, when the value of *Groups* is at its minimum (one), a runoff system has a small and negative effect, decreasing the number of parties by −0.486 (that same substantively odd result again). When the value of *Groups* is at a higher value, say, 2.5 (row 4), the impact of a runoff system is larger in magnitude and positive: in a polity with 2.5 social groups, a runoff system is estimated to increase the number of presidential candidates by (a substantively sensible) 2.523.

By looking down single columns, we see the effects of changes in the number of ethnic groups in the absence or in the presence of a runoff system. In the absence of a runoff system, a rise in the number of ethnic groups from, say, one to three coincides (oddly) with a decline in the number of presidential candidates from 3.324 to 1.366. In the presence of a runoff system, however, a rise in the number of ethnic groups from, say, one to three coincides (sensibly) with an increase in the number of presidential candidates (from 2.838 to 4.891). Subsequently, we address standard errors for these estimated changes and whether they are statistically distinguishable from zero.

Interpreting Interactive Effects Involving Different Types of Variables

Our advice on interpretation applies generally across essentially[14] all types of variables that scholars might analyze—dummy variables, discrete

14. Ordinal independent variables mildly complicate interpretation of linear-regression estimates, whether of purely linear-additive or of linear-interactive form, because linear-regression treats all independent-variable information as cardinal. In practice, researchers often assume ordinal variables to give cardinal, or close enough to cardinal,

variables, continuous variables, and so on—and any nonlinear transformations (such as $\ln(x)$ or x^2, or $\{v = 1 \text{ if } x < x_0, v = 0 \text{ if } x \geq x_0\}$). Furthermore, virtually all permutations of interactions between these types of variables are logically and empirically possible,[15] and all can be interpreted using one or both of our approaches. One need not learn different interpretational techniques for each different variable and interaction type; the preceding discussion fully suffices, as we illustrate next. (Optimal presentational efficacy will often suggest different graphical and/or tabular approaches for different applications as the next section suggests and illustrates.)

Our first empirical example illustrates one of these possible permutations: an interaction between a dummy variable (*Runoff*) and a continuous variable (*Groups*). To illustrate more of the rich range of possibilities, we now introduce some additional empirical cases.

Our second example derives from public-opinion research into partisan and gender gaps in support for social welfare (e.g., Box-Steffensmeier, De Boef, and Lin 2004; Shapiro and Mahajan 1986). Theory suggests that social-welfare attitudes derive from a set of individual-level characteristics, such as partisan orientation, ideology, gender, race, and income, and that the effect of one or more of these characteristics, such as partisanship, might depend on some other characteristic, such as gender. Partisanship is strongly related to support for social-welfare programs; for example, in the United States, Republicans are less supportive of these programs than Democrats. Gender is also strongly related to support for social-welfare programs, with females generally more supportive than males. However, if partisan and gender influences are complements or substitutes in opinion formation regarding social welfare, then the effect of partisanship among females will differ from that among males. Symmetrically, the effect of gender will differ among Republicans compared with the gender effect among Democrats. A

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information. Nominal variables complicate linear-regression interpretation similarly. For binary nominal (i.e., dummy) variables, the researcher need only remember the variable's binary nature when considering substantively meaningful ranges of or changes in those variables. Since a unit change is the only change possible, whether that dummy offers nominal, ordinal, or cardinal information does not alter the mechanics of interpretation. For nominal variables with more than two categories, increases or decreases in a variable's value do not correspond to any substantive notion of increase or decrease, and so their direct use in linear regression, again whether of purely linear-additive or of linear-interactive form, is not even approximately appropriate. For use in regression analysis, researchers would first decompose such multinomial variables into sets of binary variables, each indicating one of the categories.

15. The one exception is that a dummy variable interacted with itself just gives itself back, and so x and x^2 are identical if x is a dummy.

standard linear-interactive model like (8) would enable a test of such theoretical propositions.

We analyze such a model using data from the 2004 American National Election Studies. The dependent variable is an additive index of support for the social-welfare state.[16] The independent variables are an indicator (dummy) for gender (one if female, zero if male), an indicator for partisanship (one if Republican, zero if Democrat; with all others excluded for ease of exposition), and the interaction of those two variables.

$$Opinion = \beta_0 + \beta_F Female + \beta_R Republican + \beta_{FR} Female \times Republican + \varepsilon \quad (16)$$

The OLS results appear in table 3. Note that this analysis features an interaction between two dummy variables. Differentiation (derivatives) will produce the correct expressions for the conditional effects, but calculating differences in predicted values might make more intuitive sense given the binary nature of the variables.[17] As such, these OLS results can be easily interpreted by comparing the predicted support for the social-welfare state for each of the four categories supplied by the multiplication of the two binary variables (male Democrat, female Democrat, male Republican, and female Republican).

The predicted values in table 4 suggest that there is little difference in the social-welfare support of male and female Democrats but that a gender gap does exist in support for social welfare between male and female Republicans. The gender gap is thus contingent upon partisanship. Con-

16. We provide this very simple example for pedagogical purposes; a more fully specified model would of course be more compelling. The dependent variable is compiled from support for services and spending; government provision of jobs and a standard of living; and support for federal spending on welfare programs, social security, public schools, child care, and assistance to the poor, rescaled to range from zero (least supportive) to one (most supportive).

17. Recall that derivatives are the limit of $\Delta y/\Delta x$ as Δx approaches zero. For a dichotomous variable, this is intuitively unappealing; given that the variable takes only two discrete values, 0 and 1, Δx can only be 1, 0, or -1. However, as Greene (2003) notes, "The computation of the derivatives of the conditional mean function [i.e., the regression equation] is useful when the variable in question is continuous and often produces a reasonable approximation for a dummy variable" (676). Indeed, the differentiation method will produce the correct mathematical formula for the conditional effects of a marginal change in x, and so the only issue here involves the meaningfulness of a marginal change. For linear interactions, one can simply determine the formula for the conditional effect by differentiation and then consider only discrete changes in the conditioning variable. For nonlinear interactions, however, the amount by which the conditional effect changes as the indicator or other discrete conditioning variable increases by one will not be constant over that unit range, and so the effect of a marginal change is not as substantively interesting and the difference method is more revealing.

versely, partisanship has a larger effect among men than women; male Republicans and male Democrats are farther apart than female Republicans and female Democrats. The degree of partisan polarization is contingent upon gender. Subsequently, we address the statistical uncertainty of these estimates.

Other interactions may involve the product of two continuous variables. Our third empirical example considers the duration of parliamentary governments and features this type of interaction. The dependent variable is the average duration of governments in the post–World War II era, in months, and takes values between 11 and 45.1. We model it as a function of the postwar average number of parties in government (*NP*), which ranges from 1 to 4.3; the postwar average parliamentary support for government in the legislature (i.e., the percentage of lower house

TABLE 3. OLS Regression Results, *Support for Social Welfare*

	Coefficient (standard error) *p*-Value
Female	−0.0031
	(0.0144)
	0.828
Republican	−0.2205
	(0.0155)
	0.000
Female × *Republican*	0.0837
	(0.0214)
	0.000
Intercept	0.7451
	(0.0110)
	0.000
N (df)	1,077 (1,073)
Adjusted R^2	0.223
$P > F$	0.000

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided p-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

TABLE 4. Predicted *Support for Social Welfare*

	Democrats (*Republican* = 0)	*Republicans* (*Republican* = 1)
Males (*Female* = 0)	0.745	0.525
Females (*Female* = 1)	0.742	0.605

seats held by the governing party or parties) (*PS*), which ranges from 41.1 to 80.4; and the level of party discipline (*PD*), an indicator for high levels of party discipline.[18] We specify an interaction between the number of parties in government and the parliamentary support for government, with the idea that as the degree of support for the governing party increases, the effect of the number of parties in government on the duration of government will likely decline (and vice versa). The term *PD* serves as a control; later we expand this model to illustrate other issues. Here, we estimate the following model:

$$\begin{aligned} \textit{Government Duration} = \beta_0 + \beta_{np}NP + \beta_{ps}PS + \beta_{npps}NP \times PS \\ + \beta_{pd}PD + \varepsilon \end{aligned} \tag{17}$$

The OLS results appear in table 5. Both differentiation and differences in predicted values are useful in interpreting the results of an analysis featuring an interaction between two continuous variables. Differentiating, the effect of *NP* on the duration of governments is

$$\partial\hat{y}/\partial NP = \hat{\beta}_{np} + \hat{\beta}_{npps}PS = -31.370 + 0.468(PS)$$

The estimated coefficient $\hat{\beta}_{np} = -31.370$ suggests that the effect of the number of governing parties on government duration is -31.370 when $PS = 0$, but setting parliamentary support to zero is a substantively meaningless value, thus reinforcing our warning that coefficients are not the same as effects in the linear-interactive model. Increases in parliamentary support attenuate this negative effect of the number of parties on government duration (as hypothesized) until parliamentary support reaches a level of 67.02. At this point, the effect has reached zero: $\partial\hat{y}/\partial NP = 0$. When parliamentary support exceeds 67.02, the effect of the number of parties on government duration becomes positive. A positive effect seems substantively odd until we consider that only grand coalitions encompassing all or most of parliament would typically exceed such a high level of parliamentary support;[19] grand coalitions including more parties intuitively might indeed last longer than grand coalitions of fewer parties, which perhaps exclude some, thereby violating such coalitions' justifying principle. This example provides an interesting case where the effect of some variable x is negative in one range of z, crosses zero, and then becomes positive in another range of z. It also illustrates the importance of

18. The dummy variable *PD* reflects our own coding of party discipline in these democracies.

19. In the sample, Austrian and especially Swiss governments exceed 67 percent average parliamentary support appreciably, and governments in Luxembourg do so slightly. Reinforcing the explanation in the text, Swiss governments serve terms that are not determined by standard parliamentary processes.

considering conditional effects at substantively meaningful values in the sample.

Similarly, the effect of the degree of parliamentary support on government duration is

$$\partial\hat{y}/\partial PS = \hat{\beta}_{ps} + \hat{\beta}_{npps}NP = -0.586 + 0.468(NP)$$

The estimated effect of *PS* on government duration is negative when *NP* assumes a zero value (but this, too, is a substantively meaningless value in this example). At the meaningful minimum value of $NP = 1$, the estimated effect of *PS* is near zero, which is substantively sensible; single-party governments tend to last to term, regardless of their margin. The conditional effect of parliamentary support on government duration crosses zero when $NP = 1.25$ and is increasingly positive as *NP* increases further. Governments tend to endure longer as their parliamentary support increases, and this is especially so for multiparty governments, likely because governments of more parties are more easily fractured by events and circumstances and so have greater need of greater parliamentary support to survive the vicissitudes of coalition politics.

As table 6 exemplifies, these results can be interpreted equivalently by

TABLE 5. OLS Regression Results, *Government Duration:* Simple Linear-Interaction Model

	Coefficient (standard error) *p*-Value
Number of Parties (*NP*)	−31.370 (11.345) *0.013*
Parliamentary Support (*PS*)	−0.586 (0.454) *0.214*
Number of Parties × *Parliamentary Support* (*NP* × *PS*)	0.469 (0.186) *0.022*
Party Discipline (*PD*)	9.847 (3.204) *0.007*
Intercept	59.273 (26.455) *0.039*
N (df)	22 (17)
Adjusted R^2	0.511
$P > F$	0.002

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided *p*-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

comparing the predicted government durations at varying meaningful levels of *NP* and *PS*, while holding any other variables in the model (*PD* in this case) also at substantively meaningful values (e.g., the sample mean or mode; in this case, we hold *PD* to a value of one).

Reading down the first column of calculated values, one sees a very modestly negative (i.e., near zero) estimated effect of *PS* under single-party government; across the entire forty-point sample range of *PS*, government duration declines by only 4.71 months (from 33.05 to 28.34). However, this effect intuitively reverses sign and grows substantially as the number of governing parties increases. When governments average three parties, predicted duration increases by a substantial 32.78 months as *PS* expands from its sample minimum 40 to maximum 80 percent. Reading across each of the rows, we see the estimated effects of the number of governing parties at given levels of parliamentary support. Intuitively, increases in *NP* are associated with declines in government duration over most values of governing support (although they are associated with longer government durations at the very high values of *PS* associated with grand coalitions). Also intuitively, these deleterious effects of *NP* are greatest where parliamentary support is weakest. Subsequently, we address the statistical uncertainty of these estimates.

Our approaches for interpreting interaction terms also apply when the interacted variables have been nonlinearly transformed, such as squared terms (a special case of linear interaction where a variable in essence interacts with itself so that its effect depends on its own level), higher order polynomials, and logs. Such nonlinear transformations also render interpretation of estimated *effects* from simple examination of estimated *coefficients* very difficult and again highlight the utility of differentiation or differencing for interpreting regression analyses employing interaction terms.

Consider the case when a researcher believes that the effect of some variable, x, depends on the level of that variable x. One way to model

TABLE 6. Predicted *Government Duration*

	$NP = 1$	$NP = 2$	$NP = 3$	$NP = 4$
$PS = 40$	33.05	20.42	7.79	−4.84
$PS = 50$	31.87	23.93	15.99	8.05
$PS = 60$	30.70	27.44	24.18	20.93
$PS = 70$	29.52	30.95	32.38	33.81
$PS = 80$	28.34	34.46	40.57	46.69

Note: Predicted values are calculated at given values, setting $PD = 1$.

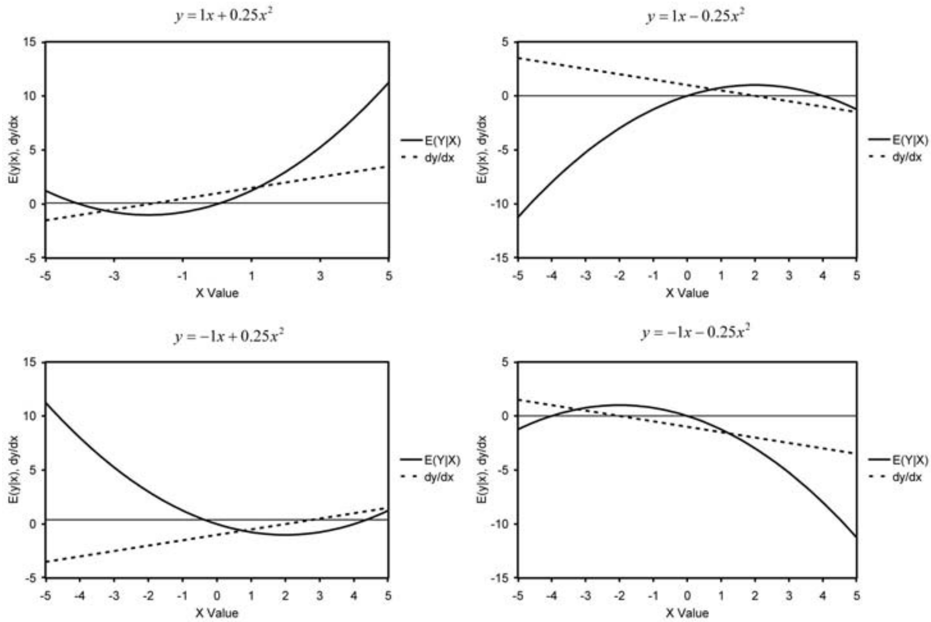

Fig. 1. Quadratic terms in linear-regression models

this proposition is to include a quadratic, or squared term, x^2, that is, the interaction of x with itself. Researchers have applied this type of interaction in several domains: the effect of age is modeled as quadratic in studies of political participation; the effect of time elapsed or remaining is modeled as quadratic in studies of the dynamics of political campaigns; loss functions in many rational-choice models take quadratic form, and so on. Generically, such quadratic models might appear as

$$y = \beta_0 + \beta_{x1}x + \beta_{x2}x^2 + \varepsilon \tag{18}$$

and specify parabolic (hump-shaped, convex or concave) relations of x and y. As always, the effect of x on y can be calculated through differentiation as

$$\partial y/\partial x = \beta_{x1} + 2\beta_{x2}x \tag{19}$$

or by differencing predicted values of y as x moves from x_a to x_c:

$$\begin{aligned}\hat{y}_c - \hat{y}_a &= \hat{\beta}_0 + \hat{\beta}_{x1}x_c + \hat{\beta}_{x2}x_c^2 - (\hat{\beta}_0 + \hat{\beta}_{x1}x_a + \hat{\beta}_{x2}x_a^2)\\ &= \hat{\beta}_{x1}(x_c - x_a) + \hat{\beta}_{x2}(x_c^2 - x_a^2)\end{aligned} \tag{20}$$

Figure 1 demonstrates how these parabolic relationships, and the associated marginal effects, look under the four possible combinations of positive and negative coefficients on the linear and quadratic terms, β_{x1} and β_{x2}, respectively, across the positive and negative value range of x.

The effect of parliamentary support on government duration, for example, might depend on the level of parliamentary support itself in this way. One might well expect an additional 10 percent support to contribute less to extending a government's duration if that increase is from 75 percent to 85 percent than if it is from 45 percent to 55 percent. Table 7 shows the estimation results of a simple model to evaluate this possibility.

Given the signs of the coefficients on *PS* and PS^2, negative and positive, respectively, and the strictly positive values of *PS*, ranging from about 40 percent to 80 percent, this example will resemble the lower left quadrant of figure 1. Figure 2 plots the estimated effect line (calculated using (19)) and predicted government duration as a function of *PS*. (Substantively, the estimated relationship seems odd and intriguing.)

Another commonly used nonlinear transformation is the natural logarithm, $\ln(x)$, which is often used when researchers want to allow the marginal effect of x to decline at higher levels of x as we have suggested here regarding the effect of parliamentary support on government duration. Common examples include the natural logs of dollars (e.g., budgetary outlays, gross domestic product [GDP], campaign spending, or personal income), of population or population density, or of elapsed time (e.g., milliseconds for response latencies or other units such as hours or

TABLE 7. OLS Regression Results, *Government Duration:* Quadratic-Term Model

	Coefficient (standard error) *p*-Value
Parliamentary Support (*PS*)	−2.734
	(2.061)
	0.200
Parliamentary Support, squared (PS^2)	0.0257
	(0.017)
	0.142
Intercept	95.20
	(62.44)
	0.144
N (df)	22 (19)
Adjusted R^2	0.158
$P > F$	0.075

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided *p*-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

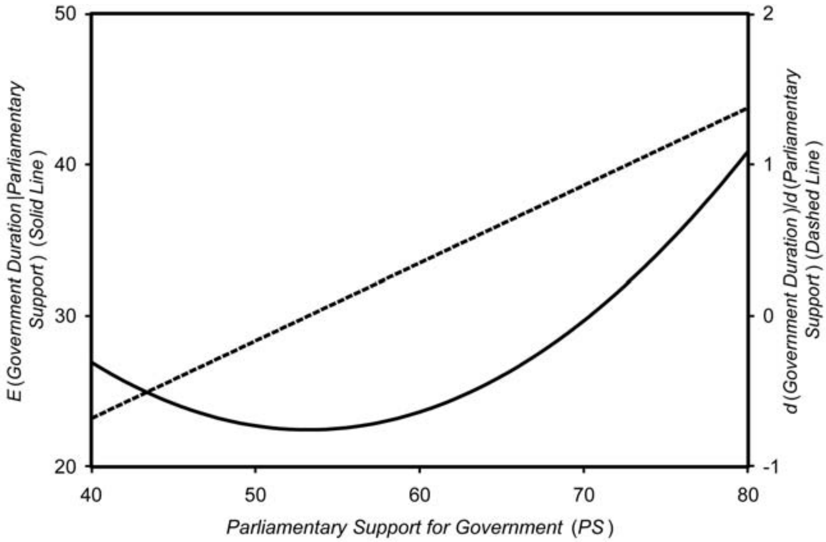

Fig. 2. Predicted *Government Duration* by *Parliamentary Support for Government,* quadratic model

days). In each of these cases, researchers will often expect the marginal effect of a unit increase in x to be greater at lower values of x and to diminish as x itself increases. In a linear-additive model, the log of x relates linearly to y, while x relates nonlinearly to y. The log transformation can also be included in the linear-interactive model. Consider, for example, a model including the natural log of parliamentary support interacted with the number of governing parties, controlling for party discipline:

$$Government\ Duration = \beta_0 + \beta_{np}NP + \beta_{\ln(ps)}\ln(PS) + \beta_{np\ln(ps)}NP \times (\ln(PS)) + \beta_{pd}PD + \varepsilon \qquad (21)$$

Table 8 gives the estimation results for this model. The slightly higher adjusted R^2 of this model and the generally greater significance of its coefficient estimates compared with the model in table 5 suggest that this model with diminishing government-duration returns to parliamentary support is somewhat superior. The effect of parliamentary support on government duration in this model can be calculated by differentiating with respect to PS: $\partial GD/\partial PS = (\beta_{\ln(ps)} + \beta_{np\ln(ps)}NP)/PS$.[20] Differentiating

20. Recall that $\partial(\ln x)/\partial x = 1/x$ and the chain rule for nested functions specifies $\partial f(g(x))/\partial x = \partial f(g)/\partial g \times \partial g(x)/\partial x$. For a model, $y = \beta_0 + \beta_{\ln(x)}\ln(x) + \beta_z z + \beta_{\ln(x)z}\ln(x) \times z$, the marginal effect of x is

$$\partial y/\partial x = (\beta_{\ln(x)} + \beta_{\ln(x)z}z)(\partial \ln(x)/\partial x) = (\beta_{\ln(x)} + \beta_{\ln(x)z}z)(1/x)$$

Please see appendix A for further description of these differentiation rules.

TABLE 8. OLS Regression Results, *Government Duration:* Log-Transformation Interactive Model

	Coefficient (standard error) p-Value
Number of Parties (*NP*)	−136.97
	(48.984)
	0.012
ln(*Parliamentary Support*) (ln(*PS*))	−43.410
	(27.417)
	0.132
Number of Parties × ln(*Parliamentary Support*) (*NP* × ln(*PS*))	32.710
	(11.956)
	0.014
Party Discipline (*PD*)	9.960
	(3.172)
	0.006
Intercept	201.41
	(111.16)
	0.088
N (df)	22 (17)
Adjusted R^2	0.520
$P > F$	0.002

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided p-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

with respect to *NP* yields its conditional effect on government duration: $\partial GD/\partial NP = \beta_{np} + \beta_{np\ln(ps)} \ln(PS)$.

Note that, befitting the diminishing returns specified for *PS*, the predicted values will vary depending upon the values of *PS* selected for the calculations. Differences in predicted values also remain straightforward to calculate.[21] Figure 3 shows one informative way to present these estimation results, plotting the predicted government duration as a function of parliamentary support at a few substantively revealing levels of the number of governing parties. (Party discipline is held fixed at one (high)

21. Specifically, the difference in predicted values of $\hat{y}$, as x increases from x_a to x_c, is

$$\begin{aligned}\hat{y}_c - \hat{y}_a &= \hat{\beta}_0 + \hat{\beta}_{\ln(x)} \ln(x_c) + \hat{\beta}_z z + \hat{\beta}_{\ln(x)z} \ln(x_c) \times z - (\hat{\beta}_0 + \hat{\beta}_{\ln(x)} \ln(x_a) + \hat{\beta}_z z \\ &\quad + \hat{\beta}_{\ln(x)z} \ln(x_a) \times z) \\ &= \hat{\beta}_{\ln(x)} \ln(x_c) - \hat{\beta}_{\ln(x)} \ln(x_a) + \hat{\beta}_{\ln(x)z} \ln(x_c) \times z - \hat{\beta}_{\ln(x)z} \ln(x_a) \times z \\ &= \hat{\beta}_{\ln(x)} \ln(x_c/x_a) + \hat{\beta}_{\ln(x)z} \ln(x_c/x_a) \times z \\ &= \ln(x_c/x_a)(\hat{\beta}_{\ln(x)} + \hat{\beta}_{\ln(x)z} z)\end{aligned}$$

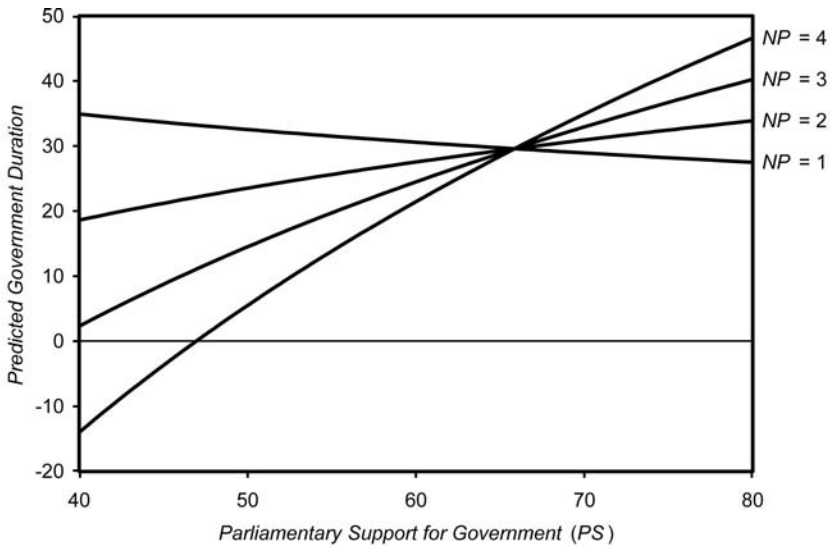

Fig. 3. Predicted *Government Duration* by *Parliamentary Support for Government,* log-transformation model

in fig. 3.) The figure reveals the essentially flat relationship between parliamentary support and government duration for single-party governments; the generally deleterious effects of the number of parties in government, especially at lower levels of parliamentary support; and the increasing effect of parliamentary support at higher numbers of governing parties. It also reveals the diminishing-returns relation of parliamentary support to government duration imposed by the log transformation. This concavity becomes more noticeable when the number of governing parties is greater, that is, when the effects of support are greater.

Threshold and spline (a.k.a. slope-shift) models represent another class of independent-variable transformations commonly used in combination with interaction terms. Suppose a researcher thought that the effect of some independent variable x changed sign or magnitude beyond some particular value, x_0. For example, the effect of years of education, *YE*, on a person's income, *Inc*, might shift at certain numbers of years representing the passing of key milestones, say, at sixteen years (typical college graduation). Up to that point, the accumulated years represent prebaccalaureate education; beyond it they represent some branch of advanced professional training. One way to specify an empirical model reflecting such a proposition would be to create a new indicator variable, call it *PB* for postbaccalaureate, equal to one if $YE \geq 16$ and zero if $YE < 16$. To allow the effect of *YE* to shift at year sixteen and above, we would want to interact YE with this transformation of itself, PB, to yield the following model:

$$Inc = \beta_0 + \beta_1 YE + \beta_2 YE \times PB + \beta_3 PB + \varepsilon \tag{22}$$

This model has a discontinuity at $YE = 16$, and so using the difference method will prove more intuitive. (In fact, the function is not differentiable at $YE = 16$.) From the difference method, then, we see exactly how the effect of income in this model of adding a year of education depends on whether that year is one of the first fifteen, the sixteenth, or beyond the sixteenth.

For values of $YE < 15$ (where an additional year of schooling would not activate the threshold of PB), a one-unit shift in schooling from YE_a to YE_c would imply a β_1 shift in income:

$$\Delta Inc = \beta_0 + \beta_1 YE_c + \beta_2 YE_c \times 0 + \beta_3 \times 0 - (\beta_0 + \beta_1 YE_a + \beta_2 YE_a \times 0 + \beta_3 \times 0)$$

$$\Delta Inc = \beta_1$$

For values of YE such that $15 \leq YE < 16$ (where the additional year of schooling activates the threshold of PB), a one-unit shift in schooling from YE_a to YE_c would imply a $\beta_1 + \beta_2 YE_c + \beta_3$ shift in income:

$$\Delta Inc = \beta_0 + \beta_1 YE_c + \beta_2 YE_c \times 1 + \beta_3 \times 1 - (\beta_0 + \beta_1 YE_a + \beta_2 YE_a \times 0 + \beta_3 \times 0)$$

$$\Delta Inc = \beta_1 + \beta_2 YE_c + \beta_3$$

For values of $YE \geq 16$ (where the additional year of schooling does not change the value of PB), a one-unit shift in schooling would imply a $\beta_1 + \beta_2$ shift in income:

$$\Delta Inc = \beta_0 + \beta_1 YE_c + \beta_2 YE_c \times 1 + \beta_3 \times 1 - (\beta_0 + \beta_1 YE_a + \beta_2 YE_a \times 1 + \beta_3 \times 1)$$

$$\Delta Inc = \beta_1 + \beta_2$$

In this slope-shift or threshold model, the prebaccalaureate piece of the income-education relation may not adjoin the postbaccalaureate piece; rather, a discontinuous jump may occur at the point. To force the segments to link continuously requires a spline model that simply regresses income on YE and $YE^* = YE - 16$ for $YE \geq 16$ and 0 otherwise. This general approach to slope-shift model specification and interpretation extends intuitively to any number of discontinuous or splined-continuous slope shifts (see Greene 2003, secs. 7.2.4–75, pp. 120–22, for further discussion).[22]

Differentiation and/or differencing thus render calculation of the estimated effects of x on y straightforward in any linear-regression model, however the independent variables may have been transformed and in whatever combinations they may interact. The section "Nonlinear Models" in chapter 5 discusses interpretation of interaction terms in nonlinear models, in which these same techniques apply.

Chained, Three-Way, and Multiple Interactions

Interactions involving more than two variables are also possible, of course, and may often be suggested theoretically. Generically, the effect of some x on y could depend on two (or more) other variables, w and z (etc.), as in this model:

$$y = \beta_0 + \beta_x x + \beta_z z + \beta_w w + \beta_{xz} xz + \beta_{xw} xw + \varepsilon \tag{23}$$

By differentiation, the effects of x, w, and z are $\partial y/\partial x = \beta_x + \beta_{xz} z + \beta_{xw} w$, $\partial y/\partial w = \beta_w + \beta_{xw} x$, and $\partial y/\partial z = \beta_z + \beta_{xz} x$, respectively. In our government-duration analysis, for example, one might well conjecture that party discipline, that is, parties' internal strategic unity, would as likely moderate the effects of the number of governing parties on government duration as would parliamentary support. A linear-interactive specification that could entertain this possibility would be

$$\begin{aligned} \textit{Government Duration} = {} & \beta_0 + \beta_{np} NP + \beta_{ps} PS + \beta_{pd} PD \\ & + \beta_{npps} NP \times PS + \beta_{nppd} NP \times PD + \varepsilon \end{aligned} \tag{24}$$

Interpretation of estimated conditional effects can once again proceed equally by differences or derivatives: $\partial GD/\partial NP = \beta_{np} + \beta_{npps} PS + \beta_{nppd} PD$, $\partial GD/\partial PS + \beta_{ps} + \beta_{npps} NP$, and $\partial GD/\partial PD = \beta_{pd} + \beta_{nppd} NP$, again safely ignoring the binary nature of PD in deriving these expressions for the conditional effects (but remembering it when considering at what values of PD or for what magnitude change in PD to calculate those conditional effects). This sort of asymmetric model, in which one variable (here NP) modifies the effects of several others (here PD and PS) or, equivalently, has its effect modified by several others (perhaps the

22. The model could equivalently be expressed as $Inc = \beta_0 + \beta_1 YE \times (1 - PB) + \beta_2 YE \times PB + \beta_3 PB + \varepsilon$. The one-unit shift at $YE < 15$ would still imply a β_1 shift in income. The one-unit shift at $15 \leq YE < 16$ would imply a $-\beta_1 YE_a + \beta_2 YE_c + \beta_3$ shift in income, and the one-unit shift at $YE \geq 16$ would imply a β_2 shift in income.

more intuitive way to express it in this substantive case), but in which those others do not condition each other's effects, might be termed a "chained-interaction" model.[23]

Substantively in this example, a model in which *NP* has its effects on government duration moderated by *PD* and *PS* certainly makes sense, but the first column of results in table 9 gives little empirical support for this chained specification, compared with the simpler model in table 5. However, we might also expect *PD* and *PS*, the missing pairwise interaction, to condition each other's government-duration effects. The durability benefits of extra seats of parliamentary support should logically depend on the reliability of those seats' votes for the government, that is, on party discipline. We call an empirical model like the one this suggests, in which the effect of each variable depends on each of the others, the complete "pairwise-interaction" model, which here just adds that one further interaction term, $PD \times PS$, to the model:

$$\begin{aligned} \textit{Government Duration} = \beta_0 &+ \beta_{np}NP + \beta_{ps}PS + \beta_{pd}PD \\ &+ \beta_{npps}NP \times PS + \beta_{nppd}NP \times PD \\ &+ \beta_{pdps}PD \times PS + \varepsilon \end{aligned} \tag{25}$$

Differentiation, as always, suffices to calculate the conditional effects in this model:

$$\frac{\partial GD}{\partial NP} = \beta_{np} + \beta_{npps}PS + \beta_{nppd}PD, \qquad \frac{\partial GD}{\partial PS} = \beta_{ps} + \beta_{npps}NP + \beta_{pdps}PD,$$

$$\frac{\partial GD}{\partial PD} = \beta_{pd} + \beta_{nppd}NP + \beta_{pdps}PS$$

Table 9 also presents the estimation results for this pairwise-interaction model, which has stronger empirical support, although the difficulty of estimating this many coefficients,[24] especially on such correlated regressors, in just twenty-two observations is also becoming evident in the standard errors of those coefficient estimates (perhaps not so much or in the same way regarding the estimated effects but we are deferring for now the discussion of the statistical certainty of conditional-effect estimates).

Finally, one might push even further along these lines to suggest that not only should the effect of each of these three factors depend on each

23. We thank an anonymous reviewer for suggesting this name for such models.

24. A model with k unique independent variables and all their pairwise interactions will comprise $(k!)/2(k-2)! + k$ regressors.

of the others in all pairwise interactions, but the effect of each might logically depend on the combination of the others present as well. For example, the government-durability benefit of additional seats of parliamentary support certainly should depend on the reliability of those seats' votes, and so we are theoretically and substantively rather confident of the $PS \times PD$ interaction. However, the impact of this "reliability adjusted" additional parliamentary support on government duration might then depend on the number of governing parties by the same logic that led us to our initial model with its single interaction term, $PS \times NP$. Table 9 also gives the estimation results for such a "fully interactive" model, which adds $PS \times NP \times PD$ to the set of pairwise interactive terms. Obviously, we are now straining the available information in the mere twenty-two observations of our example data set severely, but the empirical support

TABLE 9. OLS Regression Results, *Government Duration:* Three-Way Interactive Models

	Chained-Interaction Model	Pairwise-Interaction Model	Fully Interactive Model
Number of Parties (*NP*)	−33.810	−27.766	−51.265
	(12.013)	(11.535)	(41.342)
	0.012	*0.029*	*0.235*
Parliamentary Support (*PS*)	−0.66773	−1.5115	−2.0949
	(0.47518)	(0.61940)	(1.1699)
	0.179	*0.028*	*0.095*
Party Discipline (*PD*)	14.859	−48.690	−86.847
	(7.758)	(33.670)	(72.969)
	0.073	*0.169*	*0.254*
Number of Parties × *Parliamentary Support* (*NP* × *PS*)	0.52785	0.43443	0.84262
	(0.20651)	(0.1970)	(0.7171)
	0.021	*0.043*	*0.260*
Number of Parties × *Party Discipline* (*NP* × *PD*)	−2.6514	−3.4973	22.233
	(3.7263)	(3.4716)	(43.531)
	0.487	*0.330*	*0.617*
Party Discipline × *Parliamentary Support* (*PD* × *PS*)		1.1624	1.8219
		(0.60174)	(1.2709)
		0.073	*0.174*
Number of Parties × *Parliamentary Support* × *Party Discipline* (*NP* × *PS* × *PD*)			−0.44313
			(0.74719)
			0.563
Intercept	62.191	108.039	141.495
	(27.159)	(34.545)	(66.556)
	0.036	*0.007*	*0.052*
N (df)	22 (16)	22 (15)	22 (14)
Adjusted R^2	0.4967	0.5701	0.5507
$P > F$	0.0053	0.0031	0.0069

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided p-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

for this fully interactive model over the preceding pairwise-interactive model seems weak at any rate.[25]

Interpretation of estimated effects in such highly interactive models from coefficient estimates alone would be especially problematic. For example, the coefficient β_{np} in each of these models refers to the effect of NP when both PS and PD are zero, and the former, of course, is logically impossible. Using the derivative method allows for better interpretation:

$$\frac{\partial GD}{\partial NP} = \begin{cases} -33.81 + 0.528(PS) - 2.651(PD) \\ \qquad \text{in the chained-interaction model} \\ -27.77 + 0.434(PS) - 3.497(PD) \\ \qquad \text{in the pairwise-interaction model} \\ -51.26 + 0.843(PS) + 22.23(PD) - 0.443(PS \times PD) \\ \qquad \text{in the fully interactive model} \end{cases}$$

$$\frac{\partial GD}{\partial PS} = \begin{cases} -0.6677 + 0.528(NP) \\ \qquad \text{in the chained-interaction model} \\ -1.511 + 0.434(NP) + 1.162(PD) \\ \qquad \text{in the pairwise-interaction model} \\ -2.095 + 0.843(NP) + 1.822(PD) - 0.443(NP \times PD) \\ \qquad \text{in the fully interactive model} \end{cases}$$

$$\frac{\partial GD}{\partial PD} = \begin{cases} 14.86 - 2.651(NP) \\ \qquad \text{in the chained-interaction model} \\ -48.69 - 3.497(NP) + 1.162(PS) \\ \qquad \text{in the pairwise-interaction model} \\ -86.85 + 22.23(NP) + 1.822(PS) - 0.443(NP \times PS) \\ \qquad \text{in the fully interactive model} \end{cases}$$

The conditional effects of each independent variable in a three-way (multiple) interaction model, excepting the variables not chained in a chained-interaction model, depend on the values of two (or more) other independent variables. Accordingly, effective interpretation will require the presentation of three (or more) dimensions of information: the value of each of the conditioning variables and the estimated conditional effect corresponding to those values. The section "Presentation of Interactive Effects" in this chapter provides useful strategies for doing this.

In summary, these exercises in interpretation of coefficients should underscore the point that the variables in interactive specifications have

25. A model with k unique independent variables and all possible unique interactions of all subsets (including the whole set) of those k factors will comprise 2^k-1 regressors.

varying effects. The size and sign of the effect of x can depend critically upon the value at which the other variable, z, is held; conversely, the size and sign of the effect of z can depend critically upon the value at which the other variable, x, is held. Calling one of the coefficients involved in these effects the "main effect" and another the "interactive effect" can be quite misleading and is no substitute for understanding the model's actual estimated effects. Outside the purely linear-additive model, coefficients are not effects. Differentiation and differences of predicted values are two simple, universally applicable, and reliable tools for examining the effect of variables x and z on y in general and in interactive models in particular.

Once we have calculated these estimated conditional effects, however, we must also estimate and convey the statistical certainty of those estimates. We next discuss how to calculate standard errors for estimated conditional effects (as opposed to coefficients) and determine the degree to which these effects (as opposed to coefficients) are statistically distinguishable from zero.

Linking Statistical Tests with Interactive Hypotheses

Common social-science practice in testing interactive propositions relies almost exclusively on t-tests of significance of individual coefficients in the model. Researchers commonly compare each of the three key coefficient estimates in a typical interactive model, for example, $\hat{\beta}_x$, $\hat{\beta}_z$, and $\hat{\beta}_{xz}$ in the standard linear-interactive model, (14), to its respective standard error. Assuming that the model exhibits the necessary statistical properties otherwise (i.e., the Gauss-Markov conditions), the ratios in this comparison are t-distributed (or asymptotically normal), and so these tests are statistically valid (asymptotically). However, scholars often mistake their meaning—that is, they often mistake what these t-tests actually test—reflecting the persistent confusion of coefficients for effects and the use of the misleading main- and interactive-effect terminology. Just as the effects of variables involved in interactive terms depend upon two (or more) *coefficients* and the values of one (or more) other variable(s), so too do judgments of uncertainty surrounding those effects: their standard errors and the relevant t-statistics, confidence intervals, and hypothesis-test results (significance levels).

Single t-tests on individual coefficients on variables involved in interactive terms require care to interpret because they refer to significance at only one empirical value of the other variables. For example, β_x and β_z in our standard model (8) indicate, respectively, x's effect on y *when* z

equals zero and *z*'s effect on *y when x equals zero,* and so the standard *t*-tests on our estimates $\hat{\beta}_x$ and $\hat{\beta}_z$ indicate the significance of that variable's effect when the other variable equals zero. These specific values of zero, as noted before, may be substantively, empirically, or even logically irrelevant.

For example, in our model of the number of presidential candidates, the number of ethnic groups never equals zero in the sample and, logically, could not. Likewise in the government-duration example, neither the number of governing parties nor the level of parliamentary support could ever be zero. Thus, any inferences drawn about the statistical significance of β_R, the coefficient on *Runoff* in table 1, or of β_{np}, β_{ps}, or β_{pd} in any of the models of table 9, are largely meaningless because they refer to conditions that could not logically exist. On the other hand, inferences drawn about the statistical significance of our estimate of coefficient β_G in table 2 refer to the impact of *Groups* in the substantively meaningful case where *Runoff* equals zero. With no runoff system, the number of ethnic groups decreases the number of presidential candidates, and the test of whether the decrease is statistically significantly distinguishable from zero (i.e., no change) produces a *p*-value of 0.228.

Likewise in our model of U.S. support for social welfare (table 3), the coefficients on *Female* and *Republican* each refer to substantively important conditions. The term $\hat{\beta}_F$ is the gender gap when *Republican* equals zero, that is, among Democrats, which is substantively tiny (-0.003) and statistically indistinguishable from zero (i.e., insignificant, at $p = 0.828$), whereas $\hat{\beta}_R$ is the partisan gap when *Female* equals zero (among males), which is substantively sizable (0.22) and highly statistically distinguishable from zero ($p < 0.001$).

Even in cases like these last three, however, where individual coefficients refer to logically possible conditions that exist in the sample and, indeed, have important substantive meaning, the judgment of statistical significance is still a limited one. In the first case, it applies only to the effect of *Groups* in the absence of runoffs (*Runoff* = 0) and says nothing about that effect where runoffs occur (*Runoff* = 1). In the latter two cases, the *t*-tests on the interaction terms refer only to the significance of the gender gap among Democrats and to the partisan gap among males, and they say nothing of the other two gaps (the gender gap among Republicans and the partisan gap among females). Moreover, the specific conditions to which the coefficient estimates and their estimated standard errors refer have no greater claim than the remaining conditions do to being "main" effects in any sense.

To provide a universally valid framework for hypothesis testing of effects rather than coefficients in interactive models, consider the following types of theoretical questions often asked about them: (1) Does y depend on x, or, equivalently, is y a function of x? Does y depend on z, or, equivalently, is y a function of z? (2) Is y's dependence on x *contingent* upon or *moderated* by z, or, equivalently, does the effect of x on y depend on z? Is y's dependence on z *contingent* upon or *moderated* by x, or, equivalently, does the effect of z on y depend on x? **This is the classic interactive hypothesis; the two sets of questions are logically identical.** (3) Does y depend on x, z, and/or their interaction, xz, at all, or, equivalently, is y a function of x, z, and/or xz? In tables 10–12, we link each of these sets of theoretical questions about interactive relationships to hypotheses, and each hypothesis to its mathematical expression and to its correspondingly appropriate statistical test(s).

We start with the simpler propositions in table 10. Note that the statistical test that corresponds to each hypothesis states a null hypothesis that, as always, is what the researcher would like, theoretically, to reject statistically. The first hypothesis examines whether x has any effect on y. The mathematical expression for testing the effect of x on y includes β_x and $\beta_{xz}z$. The standard F-test on the pair of coefficients, β_x and β_{xz}, therefore identifies whether x matters (i.e., whether y depends on x). Only these coefficients both being zero would imply that y does not depend on x in any fashion in this model.

An extension of this first hypothesis would propose some direction to the effect of x on y. The "simple" extension that the effect of x on y is positive or negative is actually ill defined in linear-interactive models because the effects of x vary linearly depending on values of z, implying

TABLE 10. Does y Depend on x or z?

Hypothesis	Mathematical Expression	Statistical Test
x affects y, or y is a function of (depends on) x	$y = f(x)$ $\partial y/\partial x = \beta_x + \beta_{xz}z \neq 0$	F-test H_0: $\beta_x = \beta_{xz} = 0$
x increases y	$\partial y/\partial x = \beta_x + \beta_{xz}z > 0$	Multiple t-tests: H_0: $\beta_x + \beta_{xz}z \leq 0$
x decreases y	$\partial y/\partial x = \beta_x + \beta_{xz}z < 0$	Multiple t-tests: $\beta_x + \beta_{xz}z \geq 0$
z affects y, or y is a function of (depends on) z	$y = g(z)$ $\partial y/\partial z = \beta_z + \beta_{xz}x \neq 0$	F-test: H_0: $\beta_z = \beta_{xz} = 0$
z increases y	$\partial y/\partial z = \beta_z + \beta_{xz}x > 0$	Multiple t-tests: H_0: $\beta_z + \beta_{xz}x \leq 0$
z decreases y	$\partial y/\partial z = \beta_z + \beta_{xz}x < 0$	Multiple t-tests: H_0: $\beta_z + \beta_{xz}x \geq 0$

Note: Table assumes standard linear-interactive model, $y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz}xz + \varepsilon$, is specified.

that the effects will be positive for some, equal to (and around) zero for some, and negative for other z values, although, as stressed before, not all values of z will necessarily be substantively relevant. Accordingly, no common practice exists for testing hypotheses that x or z generally increases or decreases y in linear-interactive models because hypotheses like these are logically ambiguous in such models. Depending on where the relevant ranges of z lie, and on the accompanying standard errors, the effects could therefore be significantly positive in some meaningful ranges, significantly negative in others, and indistinguishable from zero in yet others.

To illustrate, suppose we hypothesize that x has an increasingly positive effect on y as z increases, starting from no effect at $z = 0$. Suppose also that $z < 0$ is logically impossible. In this case, even if that proposition were true and the evidence strongly supported it, the estimated effect of x on y would be zero at $z = 0$ and therefore necessarily statistically indistinguishable from zero at that point. The estimated effect also must be statistically indistinguishable from zero for some range near $z = 0$, given that all estimates have some error. (Obviously, the insignificant range will be larger the less precisely the relevant coefficients are estimated.) Therefore, hypotheses that the effects of x (or z) are generally positive or negative should instead be specified over some range of z (or x).

In stating hypotheses that prescribe the range of values of the conditioning variable(s) over which they are to be evaluated, researchers should calculate measures of uncertainty to determine whether the effects of x at several specific values of z are statistically distinguishable from zero. This approach is highlighted in the second and third hypotheses in table 10. Then, to evaluate a claim that the effect of x on y is generally positive or negative, the researcher could test whether the effect of x on y is positive over the entire logically possible, or substantively sensible, or sample range of z by conducting several t-tests along the range of z.[26] Alternatively, but equivalently, he or she could plot $\partial\hat{y}/\partial x$ over an appropriate range of z along with confidence intervals. These confidence intervals would indicate rejection of the null hypothe-

26. Alternatively, the researcher could simply estimate a linear-additive model that omits the interaction in question and test whether the coefficient on x or z significantly differs from zero in the usual manner. If the interaction truly exists, the linear-additive model would tend to produce for coefficients on x and z their *average* effect across the sample values of the other variable. If the interaction does truly exist, however, the researcher must note that this linear-additive model is misspecified, with the coefficient estimates on x and z therefore likely subject to attenuation bias and inefficiency. Accordingly, these tests would tend to be biased toward failing to reject.

sis at all values of z where zero lies outside the confidence interval around the estimated effect. However, as just explained, researchers must recognize that, in some cases, we would expect failure to reject (confidence intervals that span zero) at some levels of z even if the hypothesis generally were very strongly supported by the data.[27]

To execute this set of t-tests or generate these confidence intervals, the researcher will first need to calculate the estimated conditional effect by the differentiation or difference method. In (14), for instance, the marginal effect of x on y is $\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z$. As always, to express the uncertainty of an estimated effect, in standard errors or in confidence intervals around it, we must find its variance. It is critical to note that the coefficient estimates vary across repeated samples, not the values of z; that is, the estimated coefficients are the random variables, whereas z is fixed.[28] The estimated effect of x on y contains the product of $\hat{\beta}_{xz}$ and z; correspondingly, the estimated conditional effects will have some level of uncertainty that depends on z. Just as the effects of x on y vary with the values of z, the standard errors of the effects of x on y also vary with values of z. Each unique value in the set of estimated conditional effects (one at each value of z) will have its own variance and corresponding standard error.[29] The variance of $\partial\hat{y}/\partial x$, the estimated marginal effect of x on y, is[30]

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27. Recognizing this issue, we suggest subsequently that researchers plot the estimated effects of x across meaningful ranges of z, along with confidence intervals, and then consider the share of these confidence intervals' covered area that lies above (or below) zero as an indication of how strongly the evidence supports the proposition. Since "generally" is imprecise and involves judgment, this test is imprecise and involves judgment too, but visualizing graphically the proportion of a confidence area that lies above or below zero should help in rendering this judgment.

28. Recall that the classical linear-regression model assumes that z is fixed in repeated sampling or that, if z is stochastic, we interpret our estimates as conditioning on z (i.e., given z or holding z constant). Either way, in our estimated effects, z is fixed; $\hat{\beta}$ is what varies due to estimation uncertainty.

29. One must distinguish between the variance of the estimated marginal effect of x on y given z, $V(\partial E(y|x,z)/\partial x)$; the variance of the estimated effect of a discrete change in x on y given z, $V(\Delta E(y|x,z)/\Delta x)$; the variance of the prediction or estimate itself, $V[E(y|x,z)]$; and the variance of the prediction or forecast error, $V[y - E(y|x,z)]$. Both estimation error in $\hat{\beta}$ and the stochastic residual or error term in the model, ε, arise in the fourth case (variance of the prediction or forecast error). The variances of estimates and of estimated effects, that is, all of the other cases, involve only the estimation error in $\hat{\beta}$.

30. Given some constant c and some random variable r, $V(cr) = c^2V(r)$. Given some constant c and two random variables r_1 and r_2, the variance of the expression $V(r_1 + cr_2) = V(r_1) + c^2V(r_2) + 2cC(r_1,r_2)$. In our context, the x and z are fixed in repeated sampling, per the standard OLS assumptions, and the estimated coefficients are the random variables. More generally, for a vector of random variables, $\hat{\boldsymbol{\beta}}$, and a constant vector, $\mathbf{m}$, the variance of the linear-additive function $\mathbf{m}'\hat{\boldsymbol{\beta}}$ is $V(\mathbf{m}'\hat{\boldsymbol{\beta}}) = \mathbf{m}'\mathbf{V}(\hat{\boldsymbol{\beta}})\mathbf{m}$. Expression (26) is just one specific example of this more general formula.

$$V(\partial\hat{y}/\partial x) = V(\hat{\beta}_x) + z^2 V(\hat{\beta}_{xz}) + 2zC(\hat{\beta}_x, \hat{\beta}_{xz}) \quad (26)$$

Our uncertainty regarding the conditional effects of x on y thus depends on variability in our estimates of β_x and β_{xz}, the covariance between those estimates of β_x and β_{xz}, and the values of z at which the effects are evaluated. Our estimates of $V(\hat{\beta}_x)$ and $V(\hat{\beta}_{xz})$ are simply the squares of the standard errors of the coefficient estimates, $\hat{\beta}_x$ and $\hat{\beta}_{xz}$, reported in typical regression output. The covariance of $\hat{\beta}_x$ and $\hat{\beta}_{xz}$, however, is not typically displayed in standard regression output. It must be extracted from the estimated variance-covariance matrix of the coefficient estimates.

A variance-covariance matrix[31] is a symmetric matrix that contains the variance of each estimated coefficient along the diagonal elements and the covariance of each estimated coefficient with the other estimated coefficients in the off-diagonal elements:

$$\mathbf{V}(\hat{\boldsymbol{\beta}}) = \begin{bmatrix} V(\hat{\beta}_1) & & & \\ C(\hat{\beta}_1,\hat{\beta}_2) & V(\hat{\beta}_2) & & \\ \vdots & & \ddots & \\ C(\hat{\beta}_1,\hat{\beta}_k) & C(\hat{\beta}_2,\hat{\beta}_k) & \cdots & V(\hat{\beta}_k) \end{bmatrix}$$

In practice, we use estimates of $V(\hat{\beta}_x)$, $V(\hat{\beta}_{xz})$, and $C(\hat{\beta}_x, \hat{\beta}_{xz})$, which we will designate as $\widehat{V(\hat{\beta}_x)}$, $\widehat{V(\hat{\beta}_{xz})}$, and $\widehat{C(\hat{\beta}_x,\hat{\beta}_{xz})}$. The desired estimate of $C(\hat{\beta}_x,\hat{\beta}_{xz})$ will appear as the off-diagonal element in the estimated variance-covariance matrix that corresponds to $\hat{\beta}_x$ and $\hat{\beta}_{xz}$. In most software, researchers can easily retrieve this estimated variance-covariance matrix by a single postestimation command.[32]

To execute the tests or construct the confidence intervals suggested in the second and third rows of table 10, then, the researcher calculates the effect of x at some value of z, $\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z$, and the estimated variance around that effect at that value of z, $\widehat{V(\partial\hat{y}/\partial x)}$. The t-statistic for testing whether this estimate is statistically distinguishable from zero is then found by dividing the estimated effect $\partial\hat{y}/\partial x$ by the estimated standard error of $\partial\hat{y}/\partial x$ and evaluating the result against the t-distribution (with $n - k$ degrees of freedom, with n the number of observations and k the number of regressors, including the constant). The researcher would then repeat the process for other values across the relevant range of z to determine whether a general claim can be made about the direction of the effect.

31. In OLS, the variance-covariance matrix of the estimated coefficient vector is $s^2(\mathbf{X}'\mathbf{X})^{-1}$, where s^2 is our estimate of σ^2, the variance of ε.

32. In STATA, this command is “vce”.

As suggested earlier, however, determining whether the effect of x on y is *generally, typically,* or *on-average* positive or negative, a common component of the typical complex of interactive hypotheses, requires more precise definition of the italicized terms. If *on-average* refers to the effect at the sample-average value of z, then the single t-test of the effect of x at that value of z suffices. This value of z also gives the appropriate estimated effect and its statistical confidence for an on-average effect taken to mean the average in the sample of the effect of x.[33] If, however, one wishes to gauge the statistical certainty surrounding the hypothesis that the effect of x on y is generally or typically positive, we suggest plotting the $\partial\hat{y}/\partial x$ over the sample range of z, with confidence intervals.[34] Support for the hypotheses that $\partial y/\partial x$ is generally or typically positive or negative would correspond to most (unfortunately, no firm cutoff share exists) of this confidence interval lying appropriately above or below zero. One could quantify the share of the area covered by the confidence interval that lies above or below zero to give more precision to this analysis.[35]

Aside from these basic hypotheses that x affects y (perhaps with some sign over some range of z), researchers are also interested in whether and how the effects of x and of z on y depend on the other variable. Table 11 presents these interactive hypotheses.

Notice that the coefficient on xz directly reflects the presence, sign, and substantive magnitude of this conditioning relationship: that is, the degree to which the effects of x and z on y depend on the other variable's value. As such, the standard t-test of the coefficient on the multiplicative term tests for the presence or sign of a conditioning relationship. Since the effect of x on y is $\partial y/\partial x = \beta_x + \beta_{xz}z$, a simple t-test of the null hypothesis that $\beta_{xz} = 0$ directly evaluates whether the effect of x changes as z changes. A rejection of the null hypothesis that $\beta_{xz} = 0$ thus supports the most central interactive hypothesis: the effect of x on y varies with the level of z (and vice versa). If interactive hypotheses contain a directional

33. The first section of chapter 4 shows that this hypothesis also corresponds to the standard t-statistic reported for the coefficient on x^* in an interactive model where x and z have been mean-centered (had their sample means subtracted) to x^* and z^*.

34. The "Presentation of Interactive Effects" section in this chapter discusses how to construct confidence intervals.

35. An alternative strategy would be to estimate a different model, one without the interaction term(s), and simply evaluate the usual t-test on the appropriate coefficient, on x or on z. This alternative would reveal directly whether, on average or generally, x or z had a nonzero effect on y. However, if the true relationship really is interactive, then this alternative model is misspecified, and so these t-tests would be, at minimum, inefficient. See note 27.

TABLE 11. Is *y*'s Dependence on *x* Conditional on *z* and Vice Versa? How?

Hypothesis	Mathematical Expression	Statistical Test
The effect of x on y depends on z	$y = f(xz,\cdot)$ $\partial y/\partial x = \beta_x + \beta_{xz}z = g(z)$ $\partial(\partial y/\partial x)/\partial z = \partial^2 y/\partial x\partial z = \beta_{xz} = 0$	t-test: H_0: $\beta_{xz} = 0$
The effect of x on y increases in z	$\partial(\partial y/\partial x)/\partial z = \partial^2 y/\partial x\partial z = \beta_{xz} > 0$	t-test: H_0: $\beta_{xz} \leq 0$
The effect of x on y decreases in z	$\partial(\partial y/\partial x)/\partial z = \partial^2 y/\partial x\partial z = \beta_{xz} < 0$	t-test: H_0: $\beta_{xz} \geq 0$
The effect of z on y depends on x	$y = f(xz,\cdot)$ $\partial y/\partial z = \beta_z + \beta_{xz}x = h(x)$ $\partial(\partial y/\partial z)/\partial x = \partial^2 y/\partial z\partial x = \beta_{xz} = 0$	t-test: H_0: $\beta_{xz} = 0$
The effect of z on y increases in x	$\partial(\partial y/\partial z)/\partial x = \partial^2 y/\partial z\partial x = \beta_{xz} > 0$	t-test: H_0: $\beta_{xz} \leq 0$
The effect of z on y decreases in x	$\partial(\partial y/\partial z)/\partial x = \partial^2 y/\partial z\partial x = \beta_{xz} < 0$	t-test: H_0: $\beta_{xz} \geq 0$

Note: Table assumes standard linear-interactive model, $y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz}xz + \varepsilon$, is specified.

element—for example, the effect of x on y increases as z increases, or the effect of x on y decreases as z increases—researchers might apply one-tailed tests of the null hypothesis that $\beta_{xz} \leq 0$ or $\beta_{xz} \geq 0$. These directional hypotheses are displayed in the second and third lines of table 11.[36]

Note, also, that the mathematical expression and the statistical test for the hypothesis that x conditions the effect of z on y are identical to those for the converse that z conditions the effect of x on y. This reflects the logical symmetry of interactive propositions. If z conditions the effect of x on y, then x logically must condition the effect of z on y and in the same amount. In fact, the second three rows of table 11 simply state the logical converses of the first three rows, and so the corresponding mathematical expressions and statistical tests are identical.[37]

Finally, table 12 reveals the statistical test corresponding to the broadest sort of hypothesis one might have regarding an interactive model: that y depends in some manner, be it in a linear-additive and/or a linear-interactive way, on x and/or on z. In common language, some one or combination of x and z matters for y. This corresponds statistically, quite simply, to the F-test that all three coefficients involved in the inter-

36. Since assuming directionality in this way lowers the empirical hurdle for statistical rejection, many scholars opt more conservatively for always employing nondirectional hypotheses and two-tailed tests.

37. The order of differentiation in a cross-derivative never matters, and so this symmetry does not rely on the linear-multiplicative form specifically. In any logical proposition/mathematical model, that the effect of x depends on z implies that the effect of z depends, in identical fashion, on x: $\partial(\partial y/\partial x)/\partial z \equiv \partial(\partial y/\partial z)/\partial x$ for any function $y(x,z)$. In this case, the effect of x on y, or how y changes as x changes, is $\partial y/\partial x = \beta_x + \beta_{xz}z$. The effect of z on that effect of x on y, or how z changes the effect of x on y, is analogously $\partial(\partial y/\partial x)/\partial z = \partial(\beta_x + \beta_{xz}z)/\partial z = \beta_{xz}$. The converses for the effect of z on y and how x modifies this effect are $\partial y/\partial z = \beta_z + \beta_{xz}x$ and $\partial(\partial y/\partial z)/\partial x = \partial(\beta_z + \beta_{xz}x)/\partial x = \beta_{xz}$.

TABLE 12. Does *y* Depend on *x*, *z*, or Some Combination Thereof?

Hypothesis	Mathematical Expression	Statistical Test
y is a function of (depends on) x, z, and/or their interaction	$y = f(x,z,xz)$	F-test: H_0: $\beta_x = \beta_z = \beta_{xz} = 0$

Note: Table assumes standard linear-interactive model, $y = \gamma_0 + \beta_x x + \beta_z z + \beta_{xz} xz + \varepsilon$, is specified.

action, β_x, β_z, β_{xz}, are zero. That all three of these are zero is the only condition that would render x and z wholly irrelevant to y.

Let us walk our first empirical example through the tests outlined in tables 10–12.

First, does x affect y? Does the number of presidential candidates depend in some linear or linear-interactive way on the number of ethnic groups? An F-test of the null hypothesis that $\beta_G = 0$ and $\beta_{GR} = 0$ addresses this question. The F-test produces these results:[38] $F = 2.62$; $\text{Prob}(F_{2,12} > 2.62) = 0.1140$. Whether to reject the null hypothesis depends on the researcher's desired level of certainty. At conventional levels ($p < 0.10$, $p < 0.05$, $p < 0.01$), the researcher would not (quite) reject the null.[39]

Does z affect y? Does the number of presidential candidates depend in some linear or linear-interactive way on the presence of a runoff system? The F-test of the null hypothesis that $\beta_R = 0$ and $\beta_{GR} = 0$ yields the following results: $F = 2.96$; $\text{Prob}(F_{2,12} > 2.96) = 0.0903$, which would (barely) satisfy a $p < 0.10$ criterion but fail the stricter $p < 0.05$, $p < 0.01$ criteria.

Next, we ask whether x (generally) increases y. To answer this question, the researcher should conduct t-tests of or construct confidence intervals for the effect of x across some range of values of z (corresponding to "generally"). To conduct these t-tests, one must first calculate the standard errors associated with the given marginal effect following equation (26). Table 13 displays the estimated variance-covariance matrix from our example, which we will need for these calculations.[40]

The element in the first row and first column, 0.593, is the estimated

38. In our notation, F is the calculated F-statistic, and $\text{Prob}(F_{n,m} > F)$ is the probability, under the null, of a value greater than F in an F-distribution with n and m degrees of freedom; that is, the p-level at which the null is rejected.

39. A less strictly classical approach to hypothesis testing would simply report the p-level and leave the reader to determine how much weight to assign a result with this level of statistical significance.

40. Appendix B provides step-by-step STATA commands for conducting these types of calculations.

variance of $\hat{\beta}_G$, which is the square of its standard error from table 1: $0.770^2 \approx 0.593$. Likewise, the estimated variance of $\hat{\beta}_{GR}$ is the square of its standard error reported in table 1: $0.941^2 \approx 0.885$. The information we need from the variance-covariance matrix that we do not see in the typical regression output is $\widehat{C(\hat{\beta}_G,\hat{\beta}_{GR})}$, which is −0.593. To calculate the estimated variance of the estimated marginal effects, we simply substitute these values from the estimated variance-covariance matrix into equation (26).

$$\begin{aligned}
\widehat{V(\partial\hat{y}/\partial G)} &= \widehat{V(\hat{\beta}_G)} + Runoff^2\ \widehat{V(\hat{\beta}_{GR})} + 2 \times Runoff \\
&\quad \times\ \widehat{C(\hat{\beta}_G,\hat{\beta}_{GR})} \\
\widehat{V[(\partial\hat{y}/\partial G) \mid Runoff = 0]} &= 0.593 + 0^2 \times 0.885 + 2 \times 0 \times -0.593 \\
&= 0.593 \\
\widehat{V[(\partial\hat{y}/\partial G) \mid Runoff = 1]} &= 0.593 + 1^2 \times 0.885 + 2 \times 1 \times -0.593 \\
&= 0.292
\end{aligned}$$

The proposition that societal groups increase the number of presidential candidates corresponds to the null hypothesis: H_0: $\beta_G + \beta_{GR}\ Runoff \leq 0$. This null hypothesis can be evaluated at the two valid values of z: zero (no runoff system) and one (a runoff system). Table 14 gives these results.

With a one-tailed p-value of 0.884, we cannot reject the null hypothesis that $\beta_G + \beta_{GR} Runoff \leq 0$ when $Runoff = 0$. In systems without runoffs, a negative or null relationship between *Groups* and *Candidates* cannot be rejected. However, with a one-tailed p-value of 0.041, we *can*

TABLE 13. Estimated Variance-Covariance Matrix of Coefficient Estimates, Predicting *Number of Presidential Candidates*

	Groups	*Runoff*	*Groups* × *Runoff*	Intercept
Groups	0.593			
Runoff	0.900	2.435		
Groups × *Runoff*	−0.593	−1.377	0.885	
Intercept	−0.900	−1.509	0.900	1.509

TABLE 14. Hypothesis Tests of whether *Groups* Affects *Number of Presidential Candidates*

	$\partial\hat{y}/\partial G$	*s.e.* ($\partial\hat{y}/\partial G$)	t-Statistic	One-Tailed p-Value H_0: $\beta_G + \beta_{GR}Runoff \leq 0$	One-Tailed p-Value H_0: $\beta_G + \beta_{GR}Runoff \geq 0$	90% Confidence Interval
Runoff = 0	−0.979	0.770	−1.271	0.886	0.114	[−2.352, 0.394]
Runoff = 1	1.026	0.540	1.902	0.041	0.959	[0.064, 1.988]

reject the null hypothesis that *Groups* decrease or have no effect on *Candidates* in favor of the alternative that some positive relationship between *Groups* and *Candidates* exists when *Runoff* = 1. To test the reverse directional hypothesis, that the number of societal groups decreases the number of presidential candidates, we pose the opposite null hypothesis: $\beta_G + \beta_{GR}Runoff \geq 0$ and reevaluate. In the absence of a runoff system, the one-tailed p-value is 0.116, which actually (substantively oddly, as we have noted) approaches significance. In the presence of a runoff system, the one-tailed p-value of 0.959 suggests that we are quite unable to reject the null hypothesis of a positive or null relationship between *Groups* and *Candidates*.

To test the analogous directional hypotheses with respect to the effect of a runoff system on the number of presidential candidates, the researcher could conduct a number of t-tests over a logically relevant range of *Groups*. Table 15 displays some examples.

Here, we see that evaluation of the null hypothesis of $\beta_R + \beta_{GR}Groups \leq 0$ changes for various values of *Groups*. As the number of ethnic groups increases, our ability to reject the null hypothesis that runoff systems reduce the number of candidates increases. When *Groups* exceeds 1.5, the hypothesis test begins to approach conventional significance levels. At *Groups* = 2, we can reject the null hypothesis that runoff systems reduce the number of candidates. To investigate whether a runoff system decreases the number of presidential candidates, we reevaluate the t-statistics for the null hypothesis: $\beta_R + \beta_{GR}Groups \geq 0$. At the resulting one-tailed p-values, we cannot remotely reject the null hypothesis in any case, thus lending no support at all to the reverse proposition.

So far, then, the evidence perhaps weakly suggests that the number of ethnic groups relates to the number of presidential candidates and slightly less weakly suggests that the presence or absence of runoff systems does so. The best that might be said regarding the results for the general direction of these relationships is that the evidence suggesting that runoffs

TABLE 15. Hypothesis Tests of whether *Runoff* Affects *Number of Presidential Candidates*

	$\partial\hat{y}/\partial R$	*s.e.* $(\partial\hat{y}/\partial R)$	t-Statistic	One-Tailed p-Value H_0: $\beta_R + \beta_{GR}Groups \leq 0$	One-Tailed p-Value H_0: $\beta_R + \beta_{GR}Groups \geq 0$	90% Confidence Interval
Groups = 1	−0.486	0.752	−0.646	0.735	0.265	[−1.826, 0.854]
Groups = 1.5	0.517	0.542	0.954	0.180	0.820	[−0.449, 1.483]
Groups = 2	1.520	0.682	2.229	0.023	0.977	[0.305, 2.735]
Groups = 2.5	2.522	1.038	2.430	0.016	0.984	[0.672, 4.373]
Groups = 3	3.525	1.461	2.413	0.016	0.984	[0.922, 6.128]

and ethnic fragmentation might generally decrease the number of presidential candidates is consistently and considerably weaker than the evidence weighing in the other, the theoretically expected, direction.

Next, continuing to the tests outlined in table 11, comes the question of whether the effect of *Groups* on *Candidates* depends in some way on the presence or absence of a runoff system and vice versa. The answer to this central substantive question of interactive models emerges directly from the coefficient on the interactive term, β_{GR}, and its standard error. A two-tailed test of the null hypothesis H_0: $\beta_{GR} = 0$ yields a p-value of 0.054. Determination of "statistical significance" depends on the researcher's acceptable level of uncertainty: rejection at the $p < 0.10$ threshold, near rejection at a $p < 0.05$ threshold, and failure to reject at the tighter $p < 0.01$ level. The symmetry of interaction terms also implies the same answer for whether *Groups* modifies the effect of *Runoff*.

The directional hypothesis of whether runoffs increase the effect of *Groups* on *Candidates* requires a one-tailed test of the null H_0: $\beta_{GR} \leq 0$, which yields a p-value of 0.027. The researcher can reject the null hypothesis of a negative or nonzero coefficient in favor of the alternative hypothesis of some positive coefficient at the 0.10 and 0.05 levels but not at the 0.01 level. The positive effect of *Groups* on *Candidates* does seem larger in runoff systems, and *Runoff* has greater positive effect with a higher number of *Groups*.

Finally, consider the test in table 12: whether x and z have any effect on y in some linear or linear-interactive fashion. Here, the researcher cares whether *Groups*, *Runoff*, and/or their product affects *Candidates*. An F-test that all three coefficients are zero, H_0: $\beta_G = \beta_R = \beta_{GR} = 0$, yields the following results: $F = 2.27$, with a p-value from the $F_{3,12}$ distribution of 0.132: not overwhelming, but not surprising and perhaps not too disappointing either, given the small sample size.

We consider the remaining empirical examples more quickly. In the support for social welfare example, an F-test on the coefficients on *Female* and the interaction between *Female* and *Republican* addresses the interesting substantive question of whether gender affects support for social welfare. The results, $F = 13.08$; $\text{Prob}(F_{2,1073} > 13.08) = 0.000$, allow us to reject confidently the null hypothesis that gender has no effect on support for social welfare. Analogously, the F-test of the two coefficients on *Republican* and on the interaction of *Female* and *Republican* produces $F = 144.07$; $\text{Prob}(F_{2,1073} > 144.07) = 0.000$, allowing confident rejection of the null hypothesis that partisanship has no effect on support for social welfare. Next, we test whether the effect of gender depends on partisanship and vice versa. Recall that these calculations require access to values in the estimated variance-covariance matrix. Table 16 contains these values.

Table 17 shows that the statistical significance of the effect of gender varies sharply by partisanship: among Democrats, we can reject neither of the directional hypotheses (no statistically discernible effect of gender exists among Democrats). Among Republicans, in contrast, we can reject the null hypothesis that females are less supportive of social welfare, at $p < 0.001$. Table 18 considers the converse: whether partisanship affects support for social welfare at various values of *Female*, that is, among males and among females. Here, the null hypothesis that *Republican* increases support for social welfare is soundly rejected among both females and males: being a Republican significantly decreases support for social welfare. Finally, an F-test of all three coefficients addresses

TABLE 16. Estimated Variance-Covariance Matrix of Coefficient Estimates, Predicting *Support for Social Welfare*

	Female	*Republican*	*Female* × *Republican*	Intercept
Female	0.00021			
Republican	0.00012	0.00024		
Female × *Republican*	−0.00021	−0.00024	0.00046	
Intercept	−0.00012	−0.00012	0.00012	0.00012

TABLE 17. Hypothesis Tests of whether *Female* Affects *Support for Social Welfare*

	$\partial\hat{y}/\partial F$	*s.e.* $(\partial\hat{y}/\partial F)$	t-Statistic	One-Tailed p-Value H_0: $\beta_F + \beta_{FR}Republican \leq 0$	One-Tailed p-Value H_0: $\beta_F + \beta_{FR}Republican \geq 0$	95% Confidence Interval
Republican = 0	−0.003	0.0144	−0.218	0.586	0.414	[−0.031, 0.025]
Republican = 1	0.081	0.0158	5.109	0.000	0.999	[0.050, 0.111]

TABLE 18. Hypothesis Tests of whether *Republican* Affects *Support for Social Welfare*

	$\partial\hat{y}/\partial R$	*s.e.* $(\partial\hat{y}/\partial R)$	t-Statistic	One-Tailed p-Value H_0: $\beta_R + \beta_{FR}Female \leq 0$	One-Tailed p-Value H_0: $\beta_R + \beta_{FR}Female \geq 0$	95% Confidence Interval
Female = 0	−0.220	0.0155	−14.18	0.999	0.000	[−0.251, −0.190]
Female = 1	−0.137	0.0147	−9.33	0.999	0.000	[−0.166, −0.108]

whether partisanship or gender affect support for social welfare somehow. Here $F = 103.65$; $\text{Prob}(F_{3,1073} > 103.65) = 0.000$, and so we can confidently conclude that gender, partisanship, and/or their interaction significantly predict support for social welfare.

In our simple government-duration example (table 5), we test the hypothesis that parliamentary support for government has an effect on government duration using an *F*-test of the coefficients on *PS* and the interaction of *PS* and *NP*: $F = 6.50$; $\text{Prob}(F_{2,17} > 6.50) = 0.008$; we can confidently reject the null hypothesis of no effect. Similarly, the *F*-test of the coefficients on *NP* and $NP \times PS$ identifies whether the number of governing parties has an effect on government duration: $F = 4.87$; $\text{Prob}(F_{2,17} > 4.87) = 0.021$; we can reject the null hypothesis of no effect at conventional significance levels of $p < 0.10$ and $p < 0.05$.

The proposition that the number of governing parties decreases governmental duration must be evaluated at particular values of *PS*. The estimated variance-covariance matrix is provided in table 19. Table 20 gives the relevant calculations. When parliamentary support ranges from 40 percent to 60 percent, we can reject the null hypothesis that the number of governing parties increases governmental duration at conventional levels. However, when parliamentary support is high (at 70 percent), we

TABLE 19. Estimated Variance-Covariance Matrix of Coefficient Estimates, Predicting *Government Duration*

	Number of Parties	*Parliamentary Support*	$NP \times PS$	*Party Discipline*	Intercept
Number of Parties (*NP*)	128.712				
Parliamentary Support (*PS*)	4.564	0.206			
$NP \times PS$	−2.089	−0.078	0.035		
Party Discipline	2.980	0.080	−0.058	10.265	
Intercept	−274.906	−11.870	4.587	−10.666	699.881

TABLE 20. Hypothesis Tests of whether *Number of Parties* Affects *Government Duration*

	$\partial\hat{y}/\partial NP$	*s.e.* $(\partial\hat{y}/\partial NP)$	*t*-Statistic	One-Tailed *p*-Value H_0: $\beta_{np} + \beta_{npps}PS \leq 0$	One-Tailed *p*-Value H_0: $\beta_{np} + \beta_{npps}PS \geq 0$	90% Confidence Interval
$PS = 40$	−12.628	4.135	−3.054	0.996	0.004	[−19.822, −5.434]
$PS = 50$	−7.942	2.558	−3.104	0.997	0.003	[−12.393, −3.492]
$PS = 60$	−3.257	1.711	−1.903	0.963	0.037	[−6.233, −0.280]
$PS = 70$	1.429	2.500	0.572	0.288	0.712	[−2.920, 5.778]
$PS = 80$	6.115	4.063	1.505	0.075	0.925	[−0.954, 13.183]

can reject neither of the directional hypotheses, and when support is extremely high, we can actually weakly reject (at $p < 0.075$) the null hypothesis that the number of governing parties has the theoretically expected negative effect on government duration. Conversely for the effect of parliamentary support, table 21 shows that with only one governing party, neither directional hypothesis is rejected; greater parliamentary support might increase, decrease, or have no effect upon the duration of single-party governments. However, with multiple governing parties, the null hypothesis that parliamentary support decreases government duration is rejected. Thus, generally, parliamentary support seems to enhance government durability as expected, although we cannot reject the alternative for the case of single-party governments. Finally, the hypothesis that the number of governing parties, governing support, and/or their interaction significantly affects duration of governments can be evaluated using an F-test of all three coefficients. This F-test produces $F = 4.62$; $\text{Prob}(F_{3,17} > 4.62) = 0.015$. We can confidently reject the null hypothesis of no effect.

In table 7, we considered a simple model in which parliamentary support had a nonlinear relation to government duration, specified empirically by including PS and PS^2 as regressors. In this case, the effect of PS on government duration is $\partial GD/\partial PS = \beta_{ps} + 2\beta_{ps^2}PS$. The test that PS has some effect on government duration is the F-test of both coefficients, for which the table reports $p = 0.075$: moderate support. The test of whether this effect depends (linearly) on the level of parliamentary support itself (i.e., that the relationship of PS to government duration would be quadratic) is the standard t-test on $\hat{\beta}_{ps^2}$, which is reported in the table as giving $p = 0.142$: weak support. To gauge the significance of the estimated effect of PS at particular values of PS, we would use the following formula:

TABLE 21. Hypothesis Tests of whether *Parliamentary Support* Affects *Government Duration*

	$\partial\hat{y}/\partial PS$	*s.e.* $(\partial\hat{y}/\partial PS)$	t-Statistic	One-Tailed p-Value H_0: $\beta_{ps} + \beta_{npps}NP \leq 0$	One-Tailed p-Value H_0: $\beta_{ps} + \beta_{npps}NP \geq 0$	90% Confidence Interval
$NP = 1$	−0.118	0.293	−0.402	0.654	0.346	[−0.627, 0.392]
$NP = 2$	0.351	0.185	1.897	0.037	0.963	[0.029, 0.673]
$NP = 3$	0.820	0.228	3.587	0.001	0.999	[0.422, 1.217]
$NP = 4$	1.288	0.374	3.448	0.002	0.998	[0.638, 1.938]

$$\begin{aligned}\widehat{V\left(\frac{\partial \widehat{GD}}{PS}\right)} &= \widehat{V(\hat{\beta}_{ps} + 2\hat{\beta}_{ps^2}PS)} \\ &= \widehat{V(\hat{\beta}_{ps})} + \widehat{V(2\hat{\beta}_{ps^2}PS)} + 2\widehat{C(\hat{\beta}_{ps}, 2\hat{\beta}_{ps^2}PS)} \\ &= \widehat{V(\hat{\beta}_{ps})} + 4PS^2 \times \widehat{V(\hat{\beta}_{ps^2})} + 2PS \times 2\widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})} \\ &= \widehat{V(\hat{\beta}_{ps})} + 4PS^2 \times \widehat{V(\hat{\beta}_{ps^2})} + 4PS \times \widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})} \qquad (27)\end{aligned}$$

The relevant portion of the estimated variance-covariance matrix of these coefficient estimates is

$$\begin{array}{llll} \widehat{V(\hat{\beta}_{ps})} & \approx 4.247 & \widehat{C(\hat{\beta}_{ps^2}, \hat{\beta}_{ps})} & \approx -0.0343 \\ \widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})} & \approx -0.0343 & \widehat{V(\hat{\beta}_{ps^2})} & \approx 0.000281 \end{array}$$

So, for example, the standard error of the estimated marginal effect of PS on government duration at $PS = 55$ percent is

$$\begin{aligned} s.e.(\partial \widehat{GD}/\partial PS) &= \sqrt{4.247 + 4 \times 55^2 \times 0.000281 + 4 \times 55 \times -0.0343} \\ &\approx 0.3 \end{aligned}$$

The marginal effect at this point is $-2.73 + 2(0.0257)55 = +0.09$ and, given the associated standard error of the marginal effect, is not remotely statistically distinguishable from zero. In fact, the estimated marginal effect is insignificant in one- or two-tailed tests over about half of the sample range of PS in this model; to present the range over which the marginal effect is distinguishable from zero, we suggest calculating and plotting the confidence intervals around the effect line depicted in figure 2 (as we do in fig. 10). We provide instructions for doing so in the next section.

In table 8, we log-transformed parliamentary support before including it in an interactive model otherwise identical to that of table 5. Accordingly, testing null hypotheses that effects equal zero (i.e., testing for the existence of effects) follow that discussion exactly. The variable NP has no effect on government duration if and only if (*iff*) the coefficients on NP and $NP \times \ln(PS)$ are both zero ($F = 5.36$; $\text{Prob}(F_{2,17} > 5.36) = 0.0157$: reject); PS has no effect *iff* the coefficients on $\ln(PS)$ and $NP \times \ln(PS)$ are both zero ($F = 6.78$; $\text{Prob}(F_{2,17} > 6.78) = 0.0068$: reject); and NP and PS have no linear or linear-interactive effect *iff* all three coefficients are zero ($F = 4.81$; $\text{Prob}(F_{3,17} > 4.81) = 0.0133$: reject). The significance of the estimated marginal effects of NP at specific values of $\ln(PS)$ and the test of whether NP generally decreases government duration likewise follow the discussion from the table 5 case exactly, merely replacing PS with $\ln(PS)$. However, estimated marginal effects of PS on government duration are $\partial\widehat{GD}/\partial PS = (\hat{\beta}_{\ln(ps)} + \hat{\beta}_{np\ln(ps)}NP)/PS$, which depend on both NP and PS; so too, then, does the standard error of this estimated effect:

$$
\begin{aligned}
\widehat{V(\partial\widehat{GD}/\partial PS)} &= V\big((\hat{\beta}_{\ln(ps)} + \hat{\beta}_{np\ln(ps)}NP)/PS\big) \\
&= \widehat{V\big((\hat{\beta}_{\ln(ps)} + \hat{\beta}_{np\ln(ps)}NP)/PS\big)} \\
&= \frac{1}{PS^2}\big(\widehat{V(\hat{\beta}_{\ln(ps)})} + \widehat{V(\hat{\beta}_{np\ln(ps)}NP)} + 2\widehat{C(\hat{\beta}_{\ln(ps)},\hat{\beta}_{np\ln(ps)}NP)}\big) \\
&= \frac{1}{PS^2}\big(\widehat{V(\hat{\beta}_{\ln(ps)})} + NP^2\,\widehat{V(\hat{\beta}_{np\ln(ps)})} + 2NP \\
&\quad \times \widehat{C(\hat{\beta}_{\ln(ps)},\hat{\beta}_{np\ln(ps)})}\big) \qquad (28)
\end{aligned}
$$

We simply insert the values from the estimated variance-covariance matrix of these coefficient estimates, along with assigned values of NP and PS, into this formula for the variance of the marginal effect of PS at those values of PS and NP.[41] For example, a three-party government that increased its parliamentary support marginally from 55 percent would increase its expected duration by about a month ($\partial\widehat{GD}/\partial PS = -43.4/55 + (32.7 \times 3)/55 \approx 0.995$), with a standard error for that estimate of $[(1/55)^2\ 751.7 + (3/55)^2\ 142.95 - (2 \times 3)/55^2 \times 302.8]^{0.5} \approx 0.271$. Dividing the estimated marginal effect by the estimated standard error yields a t-statistic of $0.995/0.271 = 3.671$, implying reject at $p(t_{17} > 3.671) = 0.0019$, for the test of the null hypothesis of no effect of PS on government duration at these levels of NP and PS. As with the preceding nonlinear transformation, however, we strongly recommend graphical presentation of such estimated effects and confidence intervals and so defer further discussion.

For the chained, pairwise, and fully interactive three-way-interaction models of government duration (table 9), finally, we could follow the same sequence of common theoretical hypotheses. In doing so, notice first that we can evaluate whether one of the three independent variables affects the dependent variable by conducting an F-test of the null hypothesis that the coefficients on all of the terms involving that variable are zero. For example, the F-test that PS "matters" has a null hypothesis that β_{ps} and β_{npps} are both zero in the chained model ($F = 5.86$; $p(F_{2,16} > 5.86) = 0.012$); that β_{ps}, β_{npps}, and β_{pspd} are zero in the pairwise model

41. Here $\widehat{V(\hat{\beta}_{\ln(ps)})}$ and $\widehat{V(\hat{\beta}_{np\ln(ps)})}$ are the squares of the standard errors reported in table 8. The term $\widehat{C(\hat{\beta}_{\ln(ps)},\hat{\beta}_{np\ln(ps)})}$ is obtained by calling up the variance-covariance matrix (not shown), $\widehat{C(\hat{\beta}_{\ln(ps)},\hat{\beta}_{np\ln(ps)})} = -302.8$.

($F = 5.81$; $p(F_{3,15} > 5.81) = 0.008$); and that β_{ps}, β_{npps}, β_{pspd}, and β_{nppspd} are all zero in the fully interactive model ($F = 4.26$; $p(F_{4,14} > 4.26) = 0.018$). That the effect of *PS* depends on *NP* or *PD* in the pairwise or fully interactive models is now also a joint hypothesis in the pairwise and fully interactive models: that the coefficients β_{npps} and β_{pspd} are both zero ($F = 5.69$; $p(F_{2,15} > 5.69) = 0.015$) and that β_{npps}, β_{pspd}, and β_{nppspd} are all zero, respectively ($F = 3.75$; $p(F_{3,14} > 3.75) = 0.036$). That the effect of *PS* depends on *NP* or that the effect of *PS* depends on *PD* are both simple-hypothesis t-tests in the pairwise model, on β_{npps} or β_{nppd} ($t = 2.2$, $p(|t_{15}| > 2.2) = 0.04$; $t = 1.9$, $p(|t_{15}| > 1.9) = 0.07$), respectively, but each is a joint-hypothesis F-test of β_{npps} and β_{nppspd} or of β_{pspd} and β_{nppspd} ($F = 2.5$; $p(F_{2,14} > 2.5) = 0.12$; $F = 1.96$; $p(F_{2,14} > 1.96) = 0.18$), respectively, in the fully interactive model. The tests for the analogous hypotheses regarding how the effects of *NP* or of *PD* depend on the one other variable or the two other variables are symmetric. Finally, that some linear or linear-interactive combination of *NP*, *PS*, and/or *PD* "matters" corresponds to the F-test of the model in each case (as reported in table 9: $F = 4.68$; $p(F_{7,14} > 4.68) = 0.007$). We highly recommend graphical methods for interpreting the sign and the statistical certainty and significance of estimated effects of each variable over ranges of each of the others, as discussed in the next section.

Presentation of Interactive Effects

Hayduk and Wonnacott (1980) noted, "While the technicalities of these [interactive] procedures have received some attention . . . the proper methods for the interpretation and visual presentation of regressions containing interactions are not widely understood" (400). This section provides guidance on presentation of results from models that include interaction terms.

Mere presentation of regression coefficients and their standard errors is inadequate for the interpretation of interactive effects. As we have seen, the estimated effects of variables involved in interactive terms and the standard errors of these estimated effects vary depending on the values of the conditioning variables. Therefore, conditional effects, as best calculated by the derivative or difference method, are most effectively conveyed in tabular and graphical forms. In the political-science literature, presentations of effects that involve interactive terms now often do utilize tables or graphs that depict the effect of x on y when z equals particular values. Presentation of estimated conditional effects across a sufficiently wide and meaningful range of values of z and indication of the estimated uncertainty of these estimated conditional effects across that range are still too often lacking, however.

Many statistical-software packages can provide conditional marginal effects or predicted values as well as standard errors for these conditional estimates, typically as part of some postestimation suite of commands.[42] Further, other programs exist that will generate estimates of uncertainty around predicted values from any estimated model using simulation (King, Tomz, and Wittenberg 2000). While we have no particular qualms about such preprogrammed commands and procedures,[43] the procedures we recommend maximize the user's control over the values at which marginal effects and predicted values are calculated and, we believe, will strengthen the user's understanding and intuition in interpreting models that contain interactive terms. We strongly recommend that the user be fully conversant with the elementary mathematical foundations underlying these procedures before taking preprogrammed commands "off the shelf."[44]

Presentation of Marginal Effects

Researchers will often wish to convey to the reader how the effect of x changes over some range of z values. The estimated marginal conditional effects of x on y are the first derivative of $\hat{y}$ with respect to x: $\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z$. We will want to discuss these conditional effects of x over some substantively revealing range of z values. One such revealing range and sequence of values, which may serve as a good default, would be an evenly spaced range of values ranging from a, the sample minimum of z, to c, its sample maximum. More generally, the researcher could calculate

42. For example, in STATA, the postestimation command `lincom` will report estimates, standard errors, t-statistics, p-levels, and a 95 percent confidence interval for any linear combination of coefficients. Appendix B contains syntax that will apply `lincom` across a range of values.

43. We emphasize, however, that the researcher should verify that the uncertainty estimates produced by these procedures do not, as some unfortunately do, erroneously add stochastic error to estimation error in calculating the uncertainty of estimated effects in models with additively separable stochastic components (like linear regression).

44. This strong warning is especially important when interpreting the effects of interactive variables. Preprogrammed commands that produce marginal effects of variables of interest will likely not recognize that a set of the variables is interactive. As such, these commands may generate a marginal effect for some covariate, naively assuming that all other variables (including the interactive term!) are held constant. This ignores the central fact that the interpretation of the effect of x requires taking into account the coefficient on x, the coefficient on xz, and values of z—underscoring our point that coefficients are not effects in models including interaction terms.

the marginal effect of x on y for any set of z values of interest. Sample means, percentiles, means plus or minus one or two standard errors, and so on, are all frequently useful default points or ranges for these considerations, but substance and researchers' presentational goals should be determinate here. Using z values of particular observations—say, of some well-known, important, or illustrative case or cases in the sample—is also often a good idea. Finally, crucially, the researcher must also report the estimated certainty of the estimated conditional effects in some manner: standard errors, t-statistics, significance levels, confidence intervals. Confidence intervals are usually more effective in graphical presentation and standard errors, t-statistics, or significance levels in tables. Confidence intervals can be generated by this formula:

$$\partial\hat{y}/\partial x \pm t_{df,p} \sqrt{\widehat{V(\partial\hat{y}/\partial x)}}$$

where $t_{df,p}$ is the critical value in a t-distribution with df degrees of freedom ($df = n - k$; n is the number of observations and k the number of regressors, including the constant) for a two-sided hypothesis test at one minus the desired confidence-interval size. For example, to obtain the lower and upper bounds of a 95 percent (90 percent) confidence interval, $t_{df,p}$ should correspond to critical values for a two-sided test at the $p = 0.05$ ($p = 0.10$) level, that is, 0.025 (0.05) on each side; with large degrees of freedom, $t_{df,0.05}$ is approximately 1.96 ($t_{df,0.10} \approx 1.65$).

In our first empirical example, we calculated two sets of conditional effects. We calculated the marginal effect of *Groups* when *Runoff* equals zero and when it equals one, and we calculated the marginal effect of *Runoff* at evenly spaced values of *Groups* from one to three. To construct confidence intervals for these estimated conditional effects, we need to determine the estimated variance of these estimated effects and choose a desired confidence level. Given our small sample size, we choose to accept lower certainty and so select a 90 percent confidence interval. This interval implies a critical value of $t_{12,0.10} = 1.782$. We would thus calculate the upper bound and lower bound for the confidence intervals as

$$\text{Upper bound: } \partial\hat{y}/\partial x + 1.782 \times \sqrt{\widehat{V(\partial\hat{y}/\partial x)}}$$

$$\text{Lower bound: } \partial\hat{y}/\partial x - 1.782 \times \sqrt{\widehat{V(\partial\hat{y}/\partial x)}}$$

Note that $\widehat{V(\partial\hat{y}/\partial x)}$ is the estimated variance of the marginal effect of x on y, produced by plugging in values from the estimated variance-covariance matrix into expression (26). When evaluating the marginal effect of x at several values of z, a graphical display of marginal effects

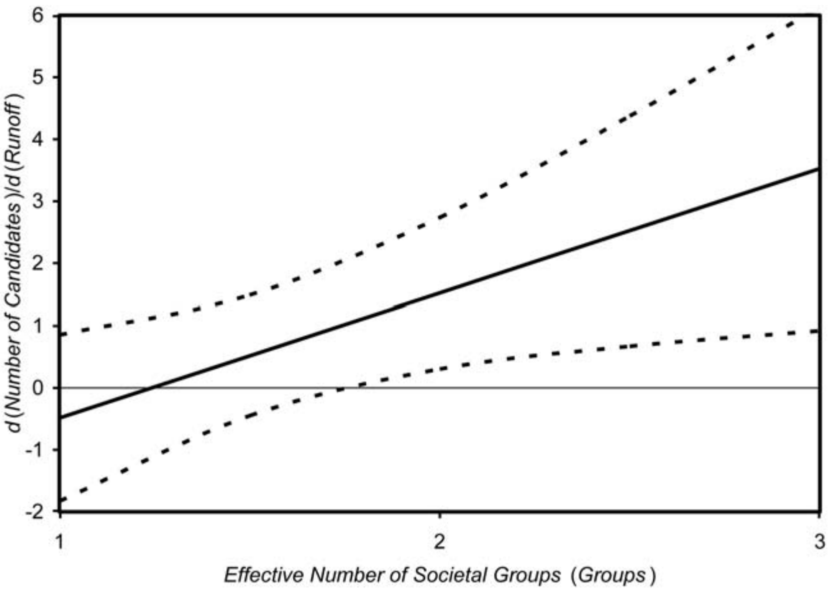

Fig. 4. Marginal effect of *Runoff*, with 90 percent confidence interval

with confidence intervals is especially effective. Figure 4, for example, displays the marginal effect of *Runoff* across a range of values of *Groups* with confidence intervals around these estimated effects. The straight line thus plots the estimated marginal conditional effects of *Runoff* as a function of *Groups*, and the confidence interval reveals the uncertainty surrounding these estimated effects. The estimated coefficient on the interaction term, $\hat{\beta}_{GR}$, gives our estimate of the slope of the marginal effect line (+2.01), indicating that the marginal effect of *Runoff* on the number of candidates is estimated to increase at a rate of about +2 *Candidates* for each one-unit increase in *Groups*. This graph shows that over the range of sample-relevant values (varying *Groups* from 1 to 3), the marginal effect of *Runoff* increases by about two for each one-unit increase in the number of groups. The marginal effect takes both negative values (though indistinguishable from zero) and positive values along the range of *Groups*. The 90 percent confidence interval overlaps zero at lower values of *Groups*, suggesting that within that range the marginal effect cannot be distinguished from zero statistically, but the confidence interval does not overlap zero when the number of societal groups exceeds 1.75.

Note how this example illustrates the ambiguity discussed previously in hypotheses of "generally positive" effects of variables involved in linear interactions. The researcher in this case would likely have hypothesized that runoff systems increase the number of presidential candidates, especially in more ethnically fragmented societies. Although the effect of runoff systems is essentially zero when fragmentation is very low, this estimated effect turns positive in even moderately fragmented societies, that is, beyond *Groups* $\approx$ 1.25, and significantly so beyond a modest *Groups* $\approx$ 1.75. Regarding the proposition that the effect of *Runoff* increases as *Groups* increases, no ambiguity arises. The marginal effect line slopes upward at the rate of $\hat{\beta}_{GR}$, and this estimated slope of the effect line is comfortably significant statistically. The ambiguity arises regarding the hypothesis of a "generally positive" effect, because the estimated effect of *Runoff* is not, in fact, positive over the entire sample range of *Groups* and is only significantly distinguishable from zero in the positive direction over some portions of that sample range. Consideration of only the coefficient on *Runoff*, $\hat{\beta}_R$, would have badly served the researcher in this example; that so-called main-effect coefficient, which actually corresponds to the logically impossible *Groups* = 0 case, is negative and larger than its standard error, yet the actual conditional effects of *Runoff* are indeed estimated to be positive over almost the entire relevant range. Graphing the estimated effects over this substantively relevant range with accompanying confidence intervals in this way reveals that this evidence actually supports that proposition reasonably strongly.

To illustrate the mathematical properties of these effect lines and their associated standard errors, imagine extending the estimated effect line from figure 4 in both directions by projecting into much lower and much higher values for *Groups*. Projecting into values of *Groups* less than 1 is substantively nonsensical, but linear regression per se imposes no such bounds on the values of independent variables, and so let us imagine that it were possible here, solely for these illustrative purposes. Calculating the estimated marginal effects of *Runoff* as the number of ethnic groups ranges from −2 to +6 produces figure 5, demonstrating several interesting properties.

As we noted before, the coefficient on *Runoff* indicates the impact of *Runoff* when *Groups* = 0, and so $\hat{\beta}_R = -2.49$ is also our estimate of the intercept of the marginal effect line (i.e., the value on the *y*-axis when *Groups* = 0), as the graph indicates. And, as evidenced in figure 4, the estimated coefficient on the interaction term, $\hat{\beta}_{GR}$, gives our estimate of the slope of the marginal effect line (+2.01), indicating that the marginal effect of *Runoff* on the number of candidates is estimated to increase at a rate of about +2 *Candidates* for each one-unit increase in *Groups*. Next, note the hourglass shape of the confidence interval around the estimated marginal-effect line; this hourglass shape is characteristic of confidence intervals for estimated conditional effects in linear-interaction models. The

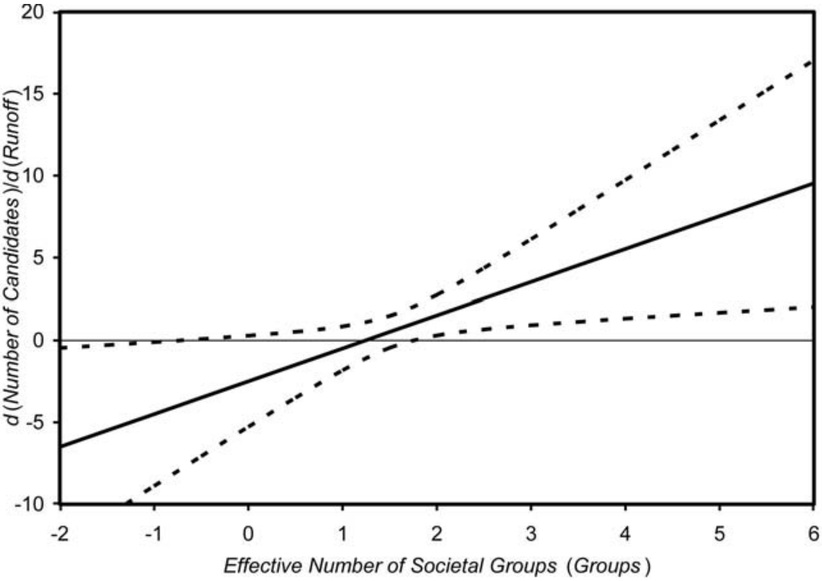

Fig. 5. Marginal effect of *Runoff,* extending the range of Groups

narrowest part of the hourglass occurs at the value of z at which there is greatest certainty concerning the size of the marginal effect of x on y. This point, intuitively, will correspond to the sample mean of the other term in the interaction (z); as always, our estimates have greatest certainty for values around the mean (*centroid* for more than one variable) of our data. The wider parts are points at which lesser certainty prevails regarding the estimated effects, which, intuitively, correspond to points farther from the mean (centroid). The characteristic hourglass shape of the confidence region results from the appearance of z^2 in the expression for the variance of the effect and also from the covariance of the coefficient estimates in that expression, which is typically negative because the corresponding variables x and xz tend to correlate positively. The relative concavity of these hourglasses generally sharpens with the magnitude of this negative correlation. In summary, the confidence intervals (regions) around conditional-effect lines will be (3D) hourglass shaped, with the narrowest points located at the mean (centroid) of the conditioning variable(s) and generally becoming more accentuated as x and xz correlate more strongly, although accentuation depends also on the relative (estimated) variances of $\hat{\beta}_R$ and $\hat{\beta}_{GR}$ and, in appearance, also on graph and z scaling.

Note also from figure 5 that the marginal effect of *Runoff* is statistically distinguishable from zero in the negative direction for values of

Groups below about −0.5, and statistically distinguishable from zero in the positive direction for values of *Groups* above about 1.75. These results illustrate clearly the following points made previously. First, the marginal effect of *Runoff* indeed varies with values of *Groups*. Second, the effect lines, being linear, will extend above and below zero for some (not necessarily meaningful) values of the variables involved. Third, our confidence regarding (i.e., standard errors and significance levels for) the marginal effect of *Runoff* also varies with values of *Groups*. Although figure 5 plots these effects and confidence intervals extending into substantively and even logically meaningless ranges, we emphasize that, in actual research, the researcher bears responsibility to ensure that interpretation and presentation of the results correspond with logically relevant and substantively meaningful values of the independent variables of interest. This implies that researchers must give such information about sample, substantive, and logical ranges necessary for the reader to recognize substantively and logically meaningful and sample-covering ranges. We have projected *Groups* into negative and very high positive values for pedagogical purposes only, to display properties of the marginal effects and confidence intervals most clearly, but we reiterate that these would *not* be logically relevant values in this case. Indeed, presenting a graph like figure 5, which extends well beyond the sample and indeed the logically permissible range, would foster misleading conclusions regarding the substantive meaning of our estimates.

Other types of graphs may more usefully depict marginal effects when conditioning variables are not continuous. For example, the variable *Runoff* takes only two values: zero in the absence of a runoff system and one in the presence of a runoff system. Accordingly, the marginal effect of *Groups* on *Candidates* is also substantively interesting for only these two values of *Runoff*. We can graph the estimated marginal effect of *Groups* on *Candidates* as a function of *Runoff*, as shown in figure 6, with 90 percent confidence intervals around each estimated marginal effect.[45] We see that in systems without runoffs, the confidence interval includes the value of zero, suggesting that the marginal effect of societal groups is not distinguishable from zero in countries with these

45. Researchers might also consider plotting normal distributions with means given by the estimated effects and standard deviations by the standard errors of those estimated effects. (Least-squares estimates are at least asymptotically normally distributed thusly.) Another option is a "box-and-whiskers" plot, with the center dots given by the estimated effects, the box around that by a confidence interval or some other multiple of the standard-error range (e.g., plus or minus one standard error), and the whiskers extending to a greater confidence interval or greater multiple of the standard-error range (plus or minus two standard errors). We prefer the simplicity of figures 6 and 7.

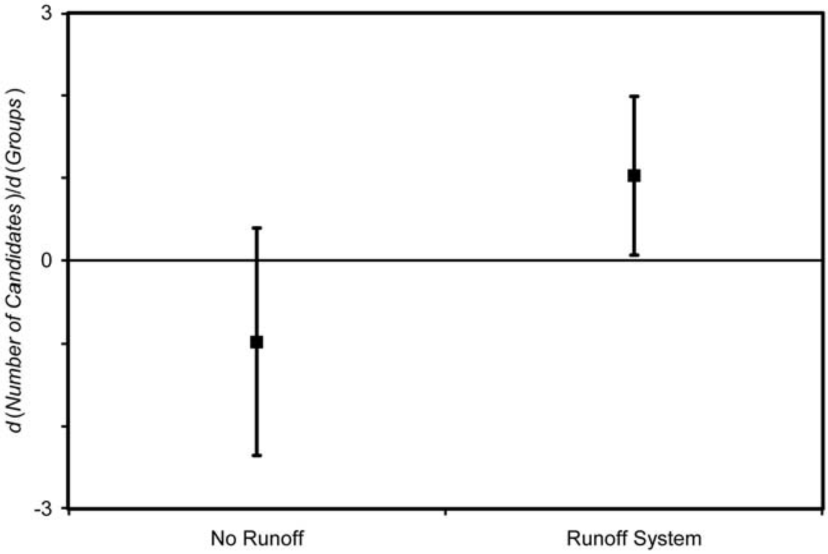

Fig. 6. Marginal effect of *Groups,* with 90 percent confidence intervals

systems. We also see that in systems with runoffs, the confidence interval does not include the value of zero; the marginal effect of societal groups can be statistically distinguished from zero in these cases. However, the confidence intervals overlap across the values of *Runoff*, suggesting that we cannot say with high levels of certainty that the marginal effects of *Groups* in cases without runoffs and with runoffs are statistically distinguishable from each other.

As another example of using this type of graph, consider our social-welfare example, where both *Female* and *Republican* are dummy variables (binary indicators) and each conditions the other's effect on support for social welfare. Thus, only four effects exist to plot: gender among Democrats and among Republicans and party among women and men. Graphically, conditional effects and associated confidence intervals in such cases are perhaps best displayed as shown in figure 7. (We adopt a more stringent confidence level, 95 percent, in these figures, given the much larger sample here.) Figure 7 reveals the estimated effects of *Female* among Democrats and Republicans with associated confidence intervals and shows the estimated effects of *Republican* for males and females, again with associated confidence intervals. In the top panel, we see that the confidence interval for the marginal effect of *Female* among Democrats includes the value of zero whereas that among Republicans does not. This graph shows that the effect of gender among Democrats does not differ statistically distinguishably from zero but the effect of gender

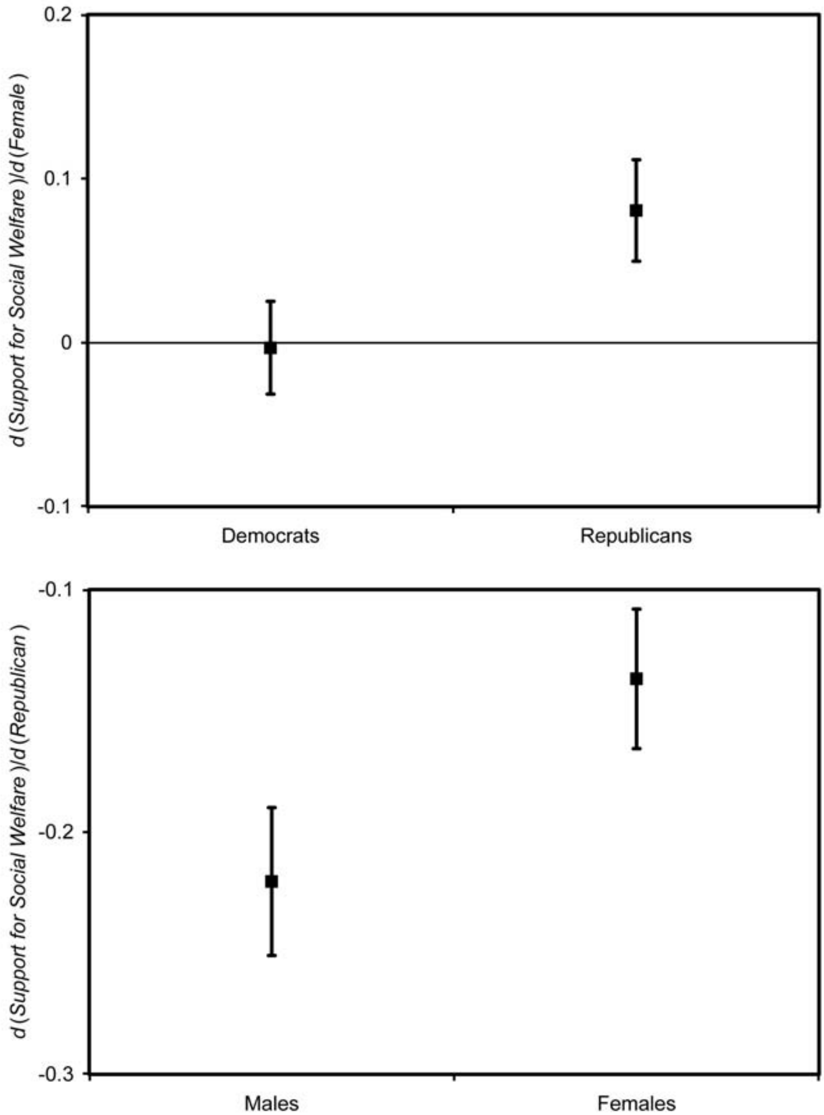

Fig. 7. Marginal effect of *Female* and *Republican,* with 95 percent confidence intervals

among Republicans does. Furthermore, the confidence intervals do not overlap, indicating that the effect of gender differs significantly between Democrats and Republicans. In the bottom panel, zero lies outside both sets of confidence intervals; the marginal effect of partisanship is significantly different from zero for both males and females. Again, the confidence intervals do not overlap, suggesting that the marginal effect of partisanship is significantly stronger (in the negative direction) among males.

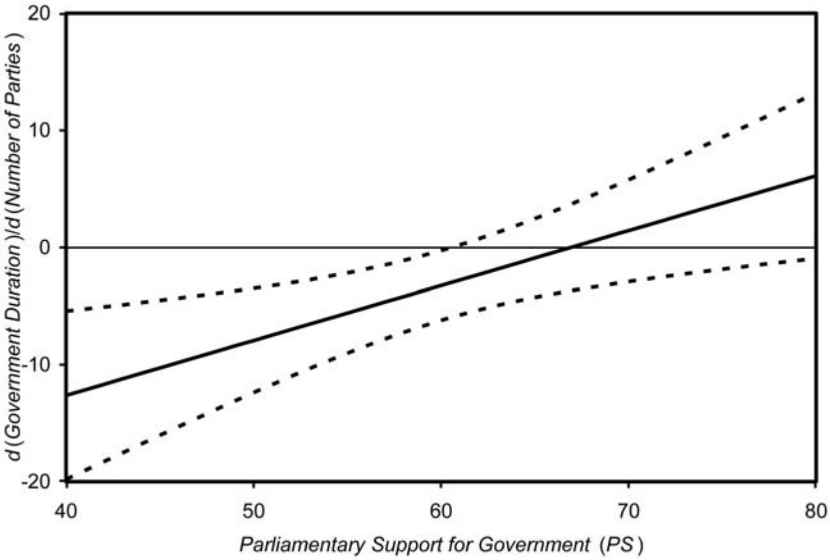

Fig. 8. Marginal effect of *Number of Parties,* with 90 percent confidence interval

Moving to our government-durability example, figures 8 and 9 illustrate the marginal effect of the number of governing parties and the marginal effect of parliamentary support on government duration from the simple, linear-interactive model of government duration featured in table 5. Figure 8 shows that the marginal effect of *NP* takes negative and positive values, depending on the value of *PS*, as we noted in that discussion. It also reveals far more clearly than discussion alone could that, at lower values of *PS*, the (negative) marginal effect of *NP* is statistically distinguishable from zero (the 90 percent confidence interval lies entirely below zero until parliamentary support reaches about 62 percent). While the estimated effect becomes positive beyond that value, it remains statistically indistinguishable from zero through the rest of the sample range. We can conclude reasonably confidently that the number of governing parties reduces government duration for parliamentary support below 62 percent, as expected, and we could note that it merely becomes statistically indistinguishable from zero beyond that, even though estimates suggest that it might even become positive. Analogously, figure 9 plots the estimated marginal effect of parliamentary support on government duration as a function of the number of governing parties. It is generally positive and becomes statistically distinguishable from zero in that direction once the number of governing parties reaches two.

Recall that figure 2 plotted estimated government duration as a quadratic function of parliamentary support. It also plotted the estimated

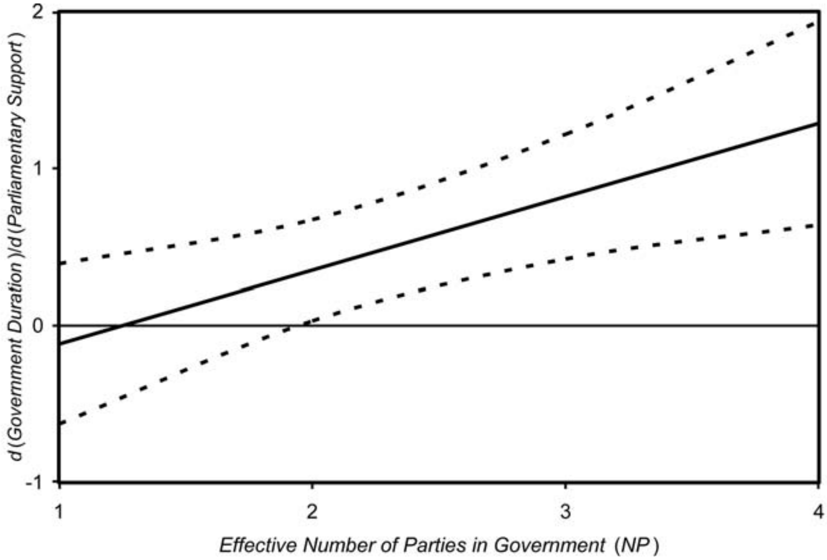

Fig. 9. Marginal effect of *Parliamentary Support for Government,* with 90 percent confidence interval

marginal effect of parliamentary support on government duration as a function of the level of support, based on the quadratic model estimated. Graphical presentation of estimates and estimated effects in nonlinear models is especially useful, and including some representation of the certainty of those estimates and estimated effects is equally crucial. Accordingly, figure 10 adds 90 percent confidence intervals to the straight line (the estimated marginal conditional effect line) in figure 2, using the square root of the expression in (27) to calculate the estimated standard error of the estimated marginal conditional effect. (We discuss construction of the confidence interval around the curved line, the predicted values, subsequently.) We take the estimated marginal effect and add (subtract) the product of the t-critical value and the estimated standard error to obtain the upper (lower) bound of the confidence interval:

$$(\hat{\beta}_{ps} + 2\hat{\beta}_{ps^2}PS) \pm 1.729 \times \left[\widehat{V(\hat{\beta}_{ps})} + 4PS^2 \times \widehat{V(\hat{\beta}_{ps^2})} + 4PS \times \widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})}\right]^{0.5}$$

Likewise, figure 3 plotted the estimated marginal nonlinear conditional effect of parliamentary support on government duration from the model specifying PS in natural log terms and interactively with the number of governing parties, NP. This presentation, too, requires indication of the uncertainty of these estimated effects. We first use the expression

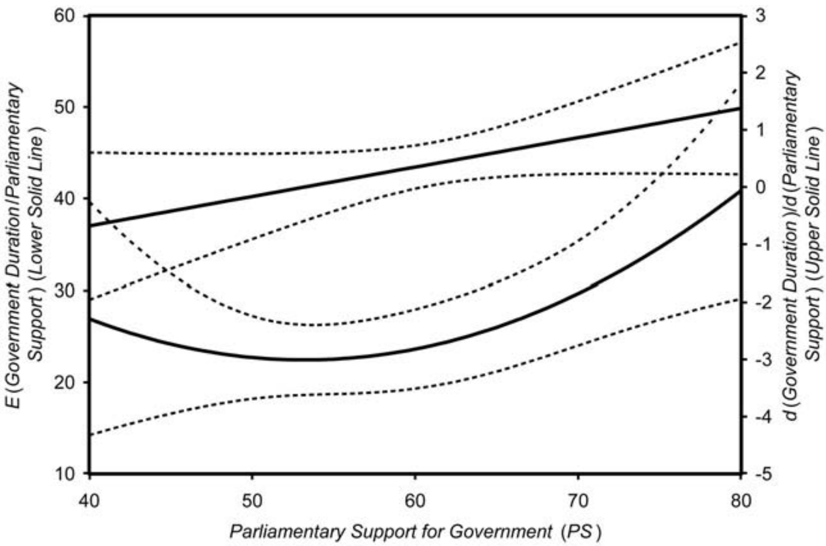

Fig. 10. Marginal effect of *Parliamentary Support* and predicted *Government Duration,* quadratic-term model, with 90 percent confidence intervals

provided in equation (28) to calculate the estimated standard error of the marginal effect of *PS* and then add (subtract) the product of the estimated standard error and the *t*-critical value to the estimated marginal effect to obtain the upper (lower) bound of the confidence interval:

$$(\hat{\beta}_{ps} + \hat{\beta}_{np\ln(ps)}NP)/PS \pm 1.74 \times \left[\frac{1}{PS^2}\left(\widehat{V(\hat{\beta}_{\ln(ps)})} + NP^2\widehat{V(\hat{\beta}_{np\ln(ps)})} + 2NP \times \widehat{C(\hat{\beta}_{\ln(ps)},\hat{\beta}_{np\ln(ps)})}\right)\right]^{0.5}$$

To accommodate the two-dimensional, monochrome technology of most print publications and to reduce visual clutter, figure 11 plots just two of these conditional-effect lines with confidence intervals, those corresponding to the revealing and interesting $NP = 2$ and $NP = 4$ cases.

The estimated marginal conditional effects of the number of governing parties on government duration can also be plotted along values of parliamentary support for government, with a confidence interval. We calculate the confidence interval as

$$(\hat{\beta}_{np} + \hat{\beta}_{np\ln(ps)}\ln(PS)) \pm 1.74 \times \left[\widehat{V(\hat{\beta}_{np})} + (\ln(PS))^2 \times \widehat{V(\hat{\beta}_{np\ln(ps)})} + 2\ln(PS) \times \widehat{C(\hat{\beta}_{np},\hat{\beta}_{np\ln(ps)})}\right]^{0.5}$$

As figure 12 reveals, the point estimate of the effect of *NP* does turn positive beyond $PS \approx 65$ percent. However, this putatively positive effect

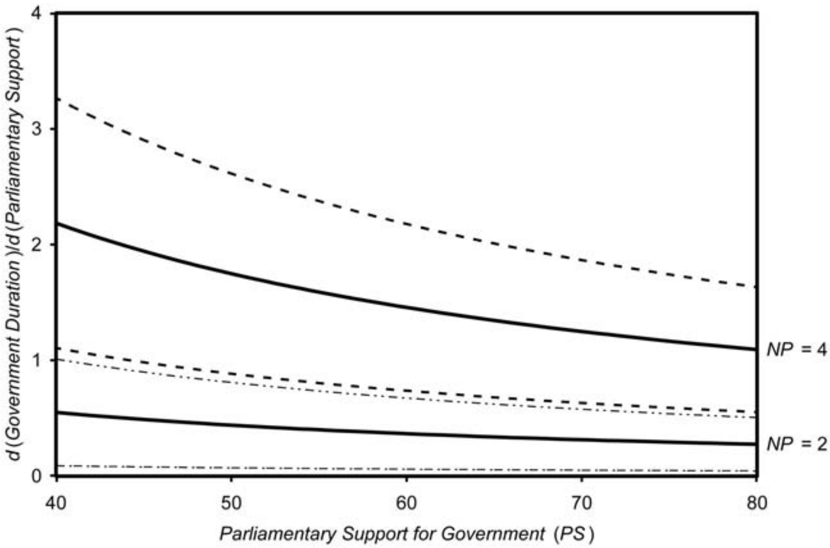

Fig. 11. Marginal effect of *Parliamentary Support for Government,* log-transformation interactive model, with 90 percent confidence intervals

never surpasses even generous levels of statistical significance ($p < 0.10$), whereas the decreasingly negative effects in the range below $PS \approx 60$ percent are statistically distinguishable from zero at this level. Thus, this fuller picture of the evidence from the empirical analysis rather suggests that, as expected intuitively, increasing government fractionalization reduces durability, but this detrimental effect generally diminishes as the strength of parliamentary support for that fractionalized government rises.

As we saw comparing the regression output from the table 5 (linear-interactive) and table 8 (log-transformed-interactive) versions of this model, the curvature of the effect lines induced by the log-transformation of *PS* is not especially strongly supported relative to a linear specification ($\bar{R}^2 = 0.520$ vs. $\bar{R}^2 = 0.511$). Graphically, this relatively weak support is seen from how easily straight conditional-effect lines could fit within the confidence intervals surrounding these slightly curved conditional-effect lines. However, we caution that exact correspondence to the significance with which the non-linear-interactive could reject the linear-interactive model does not emerge from these graphs. In fact, more generally, ability to draw flat (unconditional) effect lines within the confidence intervals of slanted (conditional) effect lines does not correspond to a hypothesis test that the effect is conditional (interactive). The correct test of that, as table 8 detailed, is the simple t-test of the interaction term in the

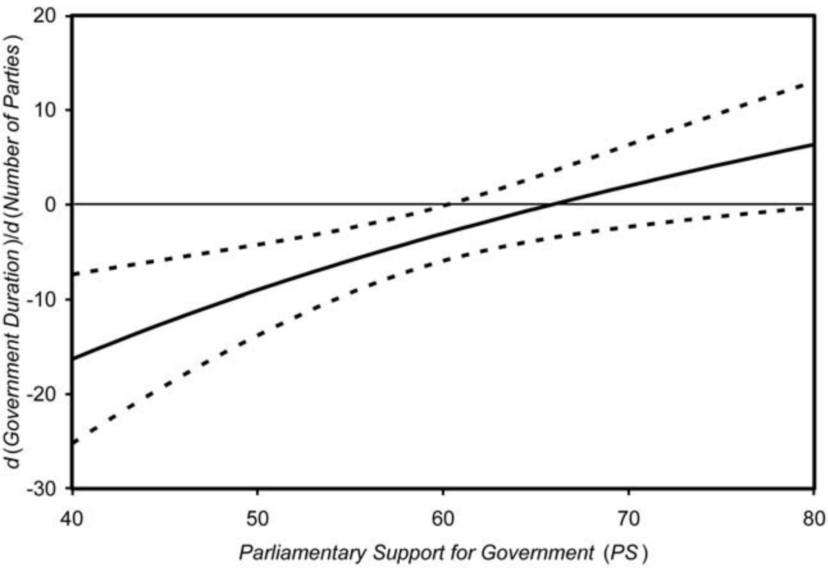

Fig. 12. Marginal effect of *Number of Parties,* log-transformation interactive model, with 90 percent confidence interval

model (or analogous *F*-tests in multiple-interaction models as in table 9). Significance of the hypothesis that the effect of x depends on z (i.e., generally) does not guarantee that the confidence intervals for the conditional effects at the high and low end of the range of z plotted or, for that matter, necessarily at any two z-values plotted, will fail to overlap.[46]

In the chained three-way-interaction model of the first column of table 9, the effects of *PS* and of party discipline, *PD*, are conditioned by one other variable, *NP*. We have already discussed and demonstrated how to present this type of conditional effect. Note, though, that the effect of *NP* in this model depends on not one but two other variables: *PS* and *PD*: $\partial\widehat{GD}/\partial NP = \hat{\beta}_{np} + \hat{\beta}_{npps}PS + \hat{\beta}_{nppd}PD$. One might consider a three-dimensional plot of such a conditional effect, plotting the marginal effect of *NP* (y-axis) as a function of *PS* (x-axis) and of *PD* (z-axis). However, conditional-effect "lines" in such cases will actually be planes plotted at linearly changing heights y as x and z change, which would be difficult to render clearly on two-dimensional pages, especially since we must also include confidence intervals, which will be (hourglass) curved

46. Indeed, in this case, the linear and the nonlinear models are nonnested and have the same degrees of freedom, and so empirical comparison of the linear versus nonlinear models must proceed on other bases entirely.

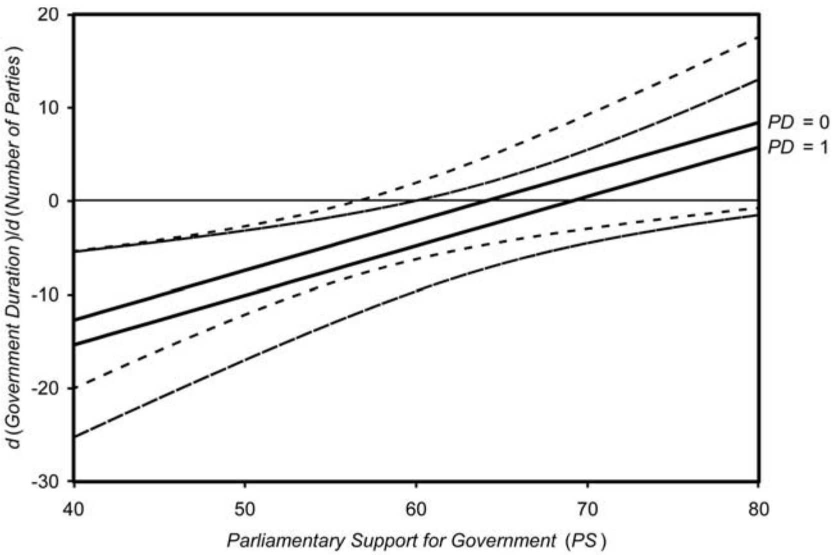

Fig. 13. Marginal effect of *Number of Parties,* chained-interaction model, with 90 percent confidence intervals

surfaces above and below that conditional-effect plane. We therefore recommend eschewing three-dimensional graphics and instead plotting contours of those three-dimensional relationships onto two dimensions. To be precise, we suggest plotting $\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z + \hat{\beta}_{xw}w$ as a function of z or w at a few values of w or z, each of which will generate one conditional-effect line, each with its own confidence interval, like those previously shown. In this case, *PD* is binary, so we could plot $\partial\widehat{GD}/\partial NP = \hat{\beta}_{np} + \hat{\beta}_{npps}PS + \hat{\beta}_{nppd}PD$ as a function of *PS* just at $PD = 0$ and at $PD = 1$, with confidence intervals, to illustrate the estimated conditional effects fully. Figure 13 demonstrates that the detrimental effect of *NP* on government durability declines with *PS*, but it does not seem to be further conditioned by *PD* in this analysis.

In the pairwise and fully interactive three-way-interaction models, finally, the effects of *NP*, *PS*, and *PD* each depend on the other two factors. Figures 14 and 15 demonstrate how researchers can graph estimation results from pairwise-interaction models effectively. Figure 14 parallels the case of figure 13, plotting how the effect of parliamentary support depends on the number of governing parties and party discipline. (The effect of *NP* symmetrically depends on *PS* and *PD* in this model, too, but those results and that figure add little to what fig. 13 already displayed.) The formulas for the effect lines in this figure parallel those from before also:

$$\begin{aligned}
\partial\widehat{GD}/\partial PS &= \hat{\beta}_{ps} + \hat{\beta}_{npps}NP + \hat{\beta}_{pspd}PD \\
\widehat{V(\partial\widehat{GD}/\partial PS)} &= \widehat{V(\hat{\beta}_{ps})} + NP^2 \times \widehat{V(\hat{\beta}_{npps})} \\
&\quad + PD^2 \times \widehat{V(\hat{\beta}_{pspd})} + 2NP \times \widehat{C(\hat{\beta}_{ps},\hat{\beta}_{npps})} \\
&\quad + 2PD \times \widehat{C(\hat{\beta}_{ps},\hat{\beta}_{pspd})} + 2NP \times PD \times \widehat{C(\hat{\beta}_{npps},\hat{\beta}_{pspd})}
\end{aligned}$$

$$90\%\ \text{c.i.:}\ \frac{\partial\widehat{GD}}{\partial PS} \pm 1.75 \times \left[\widehat{V\left(\frac{\partial\widehat{GD}}{\partial PS}\right)}\right]^{0.5}$$

The nearly nonoverlapping confidence intervals in figure 14 reveal that the effect of parliamentary support, unlike that of the number of governing parties (not shown), does seem to depend somewhat on party discipline. Intuitively, the durability-enhancing effects of larger parliamentary support are greater with higher than with lower discipline of those additional partisan supporters. The upward slopes of these conditional-effect lines show also that the benefit of greater parliamentary support to government durability seems to increase with the fractionalization of those governments. Intuitively, single-party governments can survive with bare-majority support; multiparty governments need more cushion. This feature also seems more statistically certain at higher party discipline, as the narrower confidence region for the effect at $PD = 1$ than at $PD = 0$ reveals. The effect of party discipline in these models depends on two continuous variables, NP and PS.

Therefore, three dimensions are needed to represent its conditional effects fully; however, a pair of two-dimensional graphs can suffice nearly as fully and will usually be far easier to comprehend. Namely, we recommend plotting $\partial\widehat{GD}/\partial PD = \hat{\beta}_{pd} + \hat{\beta}_{nppd}NP + \hat{\beta}_{pspd}PS$ as a function of NP at a few values of PS and as a function of PS at a few values of NP, each with confidence intervals as in figure 15.[47] The upper graph displays two flat conditional-effect lines and nearly completely nonoverlapping confidence intervals. The lower graph displays two clearly upward-sloping conditional-effect lines nearly on top of each other and with almost fully overlapping confidence intervals. These graphs suggest that the effect of

47. Using the now-familiar procedures, the estimated variance of the effect is calculated as

$$\begin{aligned}
\widehat{V\left(\frac{\partial\widehat{GD}}{\partial PD}\right)} &= \widehat{V(\hat{\beta}_{pd})} + NP^2 \times \widehat{V(\hat{\beta}_{nppd})} + PS^2 \times \widehat{V(\hat{\beta}_{pspd})} \\
&\quad + 2NP \times \widehat{C(\hat{\beta}_{pd},\hat{\beta}_{nppd})} + 2PS \times \widehat{C(\hat{\beta}_{pd},\hat{\beta}_{pspd})} + 2NP \times PS \times \widehat{C(\hat{\beta}_{nppd},\hat{\beta}_{pspd})}
\end{aligned}$$

Accordingly, the confidence interval is the estimated effect from the text plus or minus the t critical value times the square root of this expression.

party discipline on government duration seems to depend on parliamentary support but not on the number of governing parties in this model.

Figures 16 and 17 graph the estimated marginal effects of *NP* and *PD* in the fully interactive model wherein the effect of each variable depends on the values and the combination of the values of the other two variables.[48] Effective graphing techniques for fully interactive models mirror those for pairwise-interaction models because in both cases the effect of each variable depends on two others. The difference here is that when, as in figure 16, for example, plotting the marginal effect of one variable, for instance, *PS*, as a function of a second, *NP*, at different values of the third, *PD*, the marginal-effect lines will not be parallel because the effect of the first depends not just additively on the other two

48. The expressions for the estimated marginal effect of one variable in a generic three-way fully interactive model and the estimated variance of that estimated effect are

$$\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z + \hat{\beta}_{xw}w + \hat{\beta}_{xzw}zw$$

$$\begin{aligned}\widehat{V\left(\frac{\partial\hat{y}}{\partial x}\right)} &= \widehat{V(\hat{\beta}_x)} + z^2\widehat{V(\hat{\beta}_{xz})} + w^2\widehat{V(\hat{\beta}_{xw})} + z^2w^2\widehat{V(\hat{\beta}_{xzw})}\\ &\quad + 2z\widehat{C(\hat{\beta}_x,\hat{\beta}_{xz})} + 2w\widehat{C(\hat{\beta}_x,\hat{\beta}_{xw})} + 2zw\widehat{C(\hat{\beta}_x,\hat{\beta}_{xzw})}\\ &\quad + 2zw\widehat{C(\hat{\beta}_{xz},\hat{\beta}_{xw})} + 2z^2w\widehat{C(\hat{\beta}_{xz},\hat{\beta}_{xzw})} + 2zw^2\widehat{C(\hat{\beta}_{xw},\hat{\beta}_{xzw})}\end{aligned}$$

More simply, in matrix notation: $\partial\hat{y}/\partial \mathrm{x} = \mathbf{m}'\hat{\boldsymbol{\beta}} = [1\ z\ w\ zw]\begin{bmatrix}\hat{\beta}_x\\ \hat{\beta}_{xz}\\ \hat{\beta}_{xw}\\ \hat{\beta}_{xzw}\end{bmatrix}$

$$V(\mathbf{m}'\hat{\boldsymbol{\beta}}) = \mathbf{m}'\mathbf{V}(\hat{\boldsymbol{\beta}})\mathbf{m} = [1\quad z\quad w\quad zw] \times \mathbf{V}\begin{bmatrix}\hat{\beta}_x\\ \hat{\beta}_{xz}\\ \hat{\beta}_{xw}\\ \hat{\beta}_{xzw}\end{bmatrix} \times \begin{bmatrix}1\\ z\\ w\\ zw\end{bmatrix}$$

$$= [1\quad z\quad w\quad zw]\begin{bmatrix}V(\hat{\beta}_x) & C(\hat{\beta}_x,\hat{\beta}_{xz}) & C(\hat{\beta}_x,\hat{\beta}_{xw}) & C(\hat{\beta}_x,\hat{\beta}_{xzw})\\ C(\hat{\beta}_x,\hat{\beta}_{xz}) & V(\hat{\beta}_{xz}) & C(\hat{\beta}_{xz},\hat{\beta}_{xw}) & C(\hat{\beta}_{xz},\hat{\beta}_{xzw})\\ C(\hat{\beta}_x,\hat{\beta}_{xw}) & C(\hat{\beta}_{xz},\hat{\beta}_{xw}) & V(\hat{\beta}_{xw}) & C(\hat{\beta}_{xw},\hat{\beta}_{xzw})\\ C(\hat{\beta}_x,\hat{\beta}_{xzw}) & C(\hat{\beta}_{xz},\hat{\beta}_{xzw}) & C(\hat{\beta}_{xw},\hat{\beta}_{xzw}) & V(\hat{\beta}_{xzw})\end{bmatrix}\begin{bmatrix}1\\ z\\ w\\ zw\end{bmatrix}$$

In words, the variance of a sum of random variables and constants, such as an estimated conditional effect, is the sum of all the variances of the variables (the estimated coefficients implied by the conditional effect), each multiplied by the square of their cofactor (the associated independent variable(s)), plus two times each of the covariances of the variables (the estimated coefficients) times the product of their cofactors (the asociated independent variable(s)).

To complete the set of graphs, the marginal effect of *PS* can also be graphed following similar procedures.

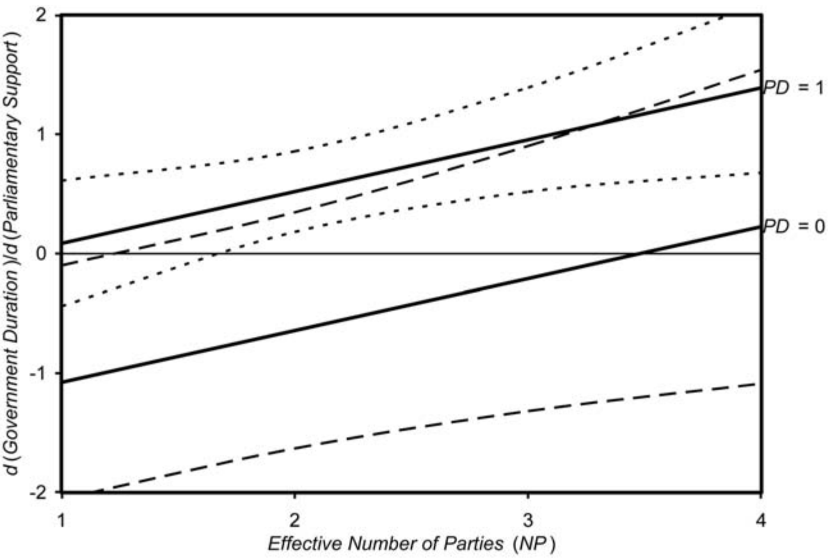

Fig. 14. Marginal effect of *Parliamentary Support for Government,* pairwise-interaction model, with 90 percent confidence intervals

but multiplicatively as well. The other major difference is the magnitude of the confidence intervals; attempting to estimate such complexly interactive relations, with seven nested, and so highly colinear, linear-interaction terms, with just twenty-two observations and fourteen degrees of freedom, will almost always prove quixotic, as it does here. We can distinguish from zero even at the low $p = 0.10$ level only (1) the intuitive increasingly beneficial effect of parliamentary support as the number of parties increases in a high party-discipline environment (fig. 16, $PD = 1$ line), (2) the converse increasingly beneficial effect of party discipline as parliamentary support for a government of relatively few parties surpasses about 50 percent (fig. 17b, $NP = 2$ line), and (3) the decreasingly beneficial effect of party discipline as the number of parties in a high parliamentary-support government rises (fig. 17a, $PS = 80$ line). Almost none of these estimated complexly conditional marginal effects is distinguishable from any other at almost any combination of independent-variable values. Researchers interested in exploring such complex context-conditionality empirically face challenges. This example illustrates the importance of maximizing observations and degrees of freedom and of leveraging theory to specify interactive hypotheses as precisely as possible (as strongly urged in chap. 2 and as demonstrated in Franzese 1999, 2002, 2003a).

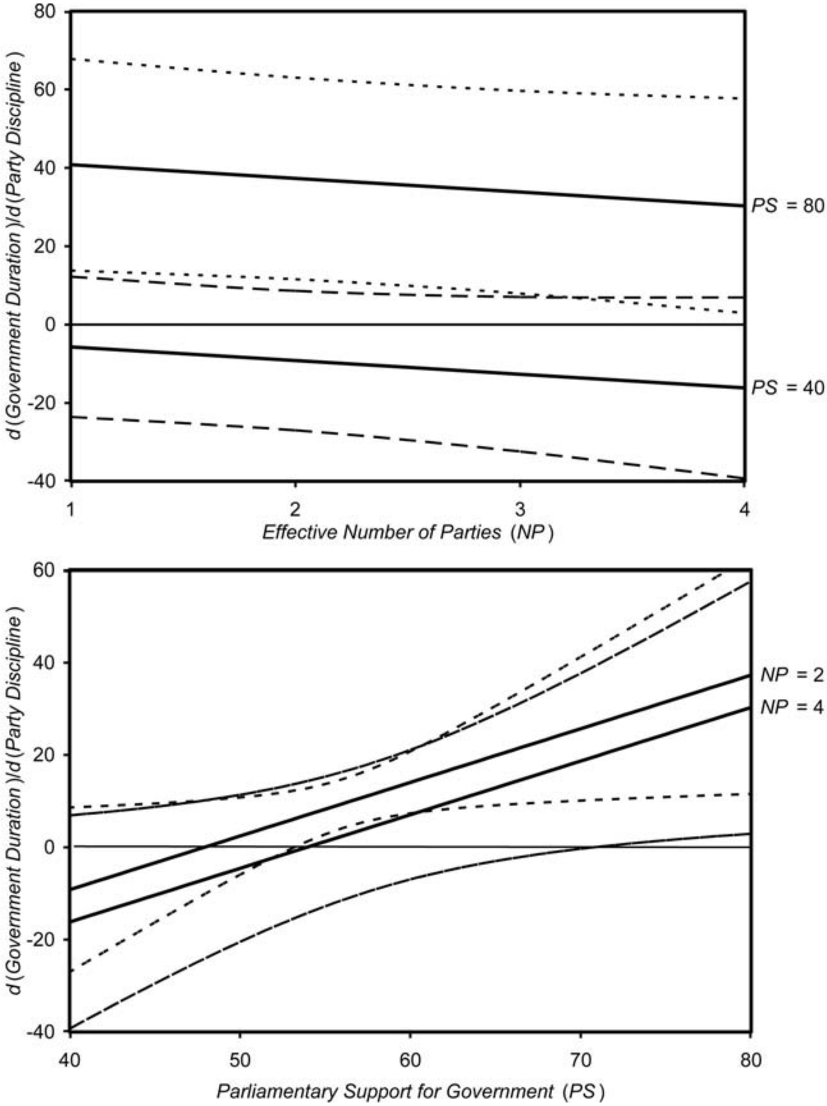

Fig. 15. Marginal effect of *Party Discipline,* pairwise-interaction model, with 90 percent confidence intervals

Presentation of Predicted Values

Aside from presenting conditional effects, researchers may also wish to present the predictions of y as x varies across a range of values, say, from x_a to x_c, its sample minimum to maximum, while holding z constant at some (meaningful and revealing) value. Changes in these predictions from

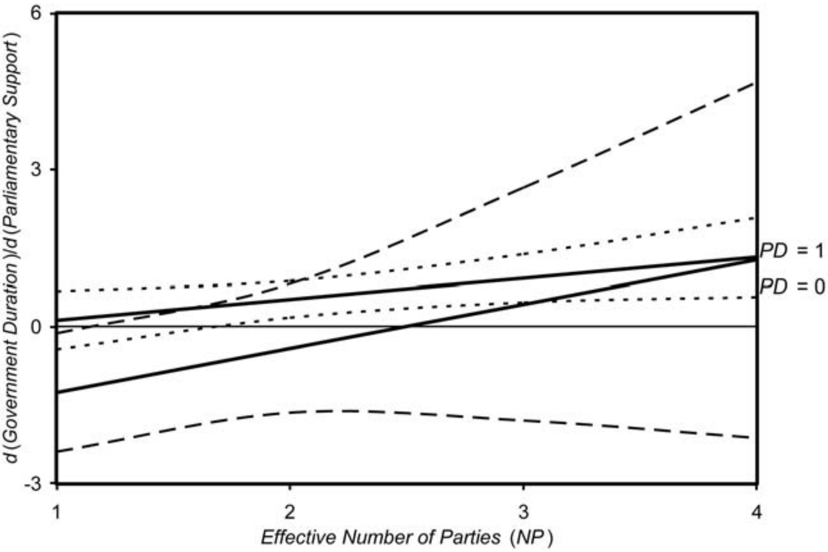

Fig. 16. Marginal effect of *Parliamentary Support for Government,* fully interactive model, with 90 percent confidence intervals

some particular $\hat{y}|x_a$ to $\hat{y}|x_c$ would reveal the effects of such changes in x on y at that level of z as just discussed, but we may also wish to present tables or graphs of predictions per se as x varies, holding z fixed. (Recall that xz will also vary with x, even though z is held constant.) Including measures of uncertainty around these predictions is again imperative, and, as with effects, each predicted value at some particular x and z values has its own level of uncertainty attached to it. Thus, tables and graphs of predicted values should also include standard errors and/or confidence intervals (variances, standard errors, significance levels) around each of those predicted values.

In the standard linear-interaction model, the variance around each predicted value is

$$V(\hat{y} \mid x,z) = V(\hat{\gamma}_0 + \hat{\beta}_x x + \hat{\beta}_z z + \hat{\beta}_{xz} xz) \tag{29}$$

Expanding this expression:[49]

$$\begin{aligned} V(\hat{y}) = V(\hat{\gamma}_0) &+ x^2 V(\hat{\beta}_x) + z^2 V(\hat{\beta}_z) + (xz)^2 V(\hat{\beta}_{xz}) \\ &+ 2xC(\hat{\gamma}_0, \hat{\beta}_x) + 2zC(\hat{\gamma}_0,\hat{\beta}_z) + 2xzC(\hat{\gamma}_0,\hat{\beta}_{xz}) \\ &+ 2xzC(\hat{\beta}_x,\hat{\beta}_z) + 2x(xz)C(\hat{\beta}_x,\hat{\beta}_{xz}) + 2z(xz)C(\hat{\beta}_z,\hat{\beta}_{xz}) \end{aligned} \tag{30}$$

49. Note 30 gives the more general linear-algebraic formula for variances of linear combinations of random variables and constants.

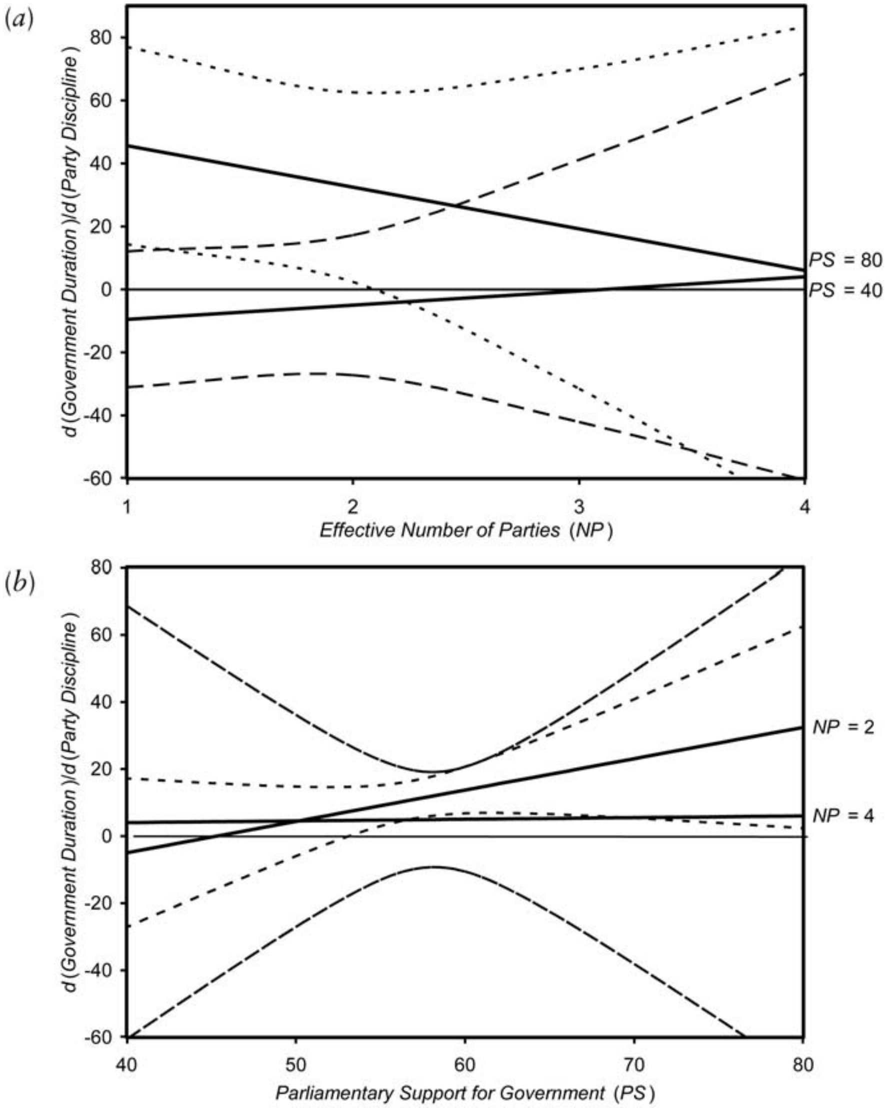

Fig. 17. Marginal effect of *Party Discipline,* fully interactive model, with 90 percent confidence intervals

In words, the variance of a sum equals the sum of the variances plus two times all the covariances. More completely, the variance of a sum of random variables (here, the coefficient estimates) times constants (here, independent variables) is equal to the sum of the variances times the associated constants squared plus two times all the covariances times the product of their constant cofactors.[50] As before, we will need the estimated variance-covariance matrix of the parameter estimates ($\mathbf{V}(\hat{\boldsymbol{\beta}})$) to calculate this, which can be easily recalled by an additional postestimation command in most statistical software.[51]

Let us use our first empirical example to calculate the predicted number of presidential candidates corresponding with various values of *Ethnic Groups* and *Runoff* along with the variance of each predicted value. Table 13 gave the variance-covariance matrix of the coefficient estimates from this model. When *Groups* = 1 and *Runoff* = 0, we predict the number of candidates to be

$$(\widehat{Candidates} \mid Groups = 1, Runoff = 0) = 4.303 - 0.979 \times 1 - 2.491 \times 0 + 2.005 \times 1 \times 0 = 3.324$$

Using equation (30), substituting *Groups* = 1 and *Runoff* = 0, yields the following expression:

$$\begin{aligned}\widehat{V(\widehat{Candidates} \mid Groups = 1, Runoff = 0)} &= \widehat{V(\hat{\beta}_0)} + 1^2\widehat{V(\hat{\beta}_G)} \\ &+ 0^2\widehat{V(\hat{\beta}_R)} + (0)(1)^2\,\widehat{V(\hat{\beta}_{GR})} \\ &+ 2 \times 1 \times \widehat{C(\hat{\beta}_0,\hat{\beta}_G)} \\ &+ 2 \times 0 \times \widehat{C(\hat{\beta}_0,\hat{\beta}_R)} \\ &+ 2 \times 1 \times 0 \times \widehat{C(\hat{\beta}_0,\hat{\beta}_{GR})} \\ &+ 2 \times 1 \times 0 \times \widehat{C(\hat{\beta}_G,\hat{\beta}_R)} \\ &+ 2 \times 1 \times (1 \times 0)\,\widehat{C(\hat{\beta}_G,\hat{\beta}_{GR})} \\ &+ 2 \times 0 \times (1 \times 0)\,\widehat{C(\hat{\beta}_R,\hat{\beta}_{GR})}\end{aligned}$$

Substituting the estimated values of the variances and covariances of the coefficients:

$$\widehat{V(\widehat{Candidates} \mid Groups = 1, Runoff = 0)} = 1.509 + 0.593 + 2 \times (-0.900) = 0.302$$

50. These are variances and confidence intervals for $E(y|x,z = z_0)$ and not forecast or prediction errors, which would include also some uncertainty due to the variance of the regression's error term. See note 30.

51. Appendix B provides step-by-step STATA commands.

Table 22 presents the standard errors of each of the predicted values as *Ethnic Groups* ranges from one to three, when *Runoff* takes the values of zero and one. These predictions can also be graphed as described later.

Obviously, these calculations will become quite cumbersome, quite quickly, in the presence of additional covariates. In fact, calculation of the variance of predicted values requires attention to the levels of all the independent variables and to the variance of each estimated coefficient and the covariances between each of the estimated coefficients. In our simple model, which includes just three variables plus an intercept, this involves ten terms. Adding just one more regressor (which did not interact with any others) would require us to include five more terms in equation (30)!

One way to simplify the expression is to use matrix algebra to depict $\hat{y}$ and to calculate $\widehat{V(\hat{y})}$ (see note 30). Note that a predicted value, $\hat{y}$, sums the products of sets of values of the right-hand-side variables and their corresponding coefficients. Let $\mathbf{M_h}$ be a j-by-k matrix of values at which x, z, and any other variables of interest in the equation are set, where j refers to the number of values at which the predicted value is calculated and k refers to the number of regressors, including the constant. Suppose we were to hold z (and any of the other variables) at some logically relevant value(s), say, z_0, and examine the predicted values of $\hat{y}$ at a set of j evenly spaced values of x from x_a to x_c and correspondingly, as xz takes j evenly spaced values from $x_a z_0$ to $x_c z_0$. In our standard equation, we have estimated coefficients for x, z, and xz, in addition to an intercept. Matrix $\mathbf{M_h}$ is thus

$$\mathbf{M_h} = \begin{bmatrix} x_a & z_0 & x_a z_0 & 1 \\ x_{a+1} & z_0 & x_{a+1} z_0 & 1 \\ \vdots & \vdots & \vdots & 1 \\ x_c & z_0 & x_c z_0 & 1 \end{bmatrix}$$

In $\mathbf{M_h}$, the value of x increments evenly from some value x_a to some other value x_c; z is fixed at z_0; and the interaction term xz varies as x does. The

TABLE 22. Confidence Intervals for Predicted *Number of Presidential Candidates*

	Runoff = 0			*Runoff* = 1		
	$\hat{y}$	$s.e.(\hat{y})$	90% Confidence Interval	$\hat{y}$	$s.e.(\hat{y})$	90% Confidence Interval
Groups = 1	3.324	0.550	[2.344, 4.305]	2.838	0.512	[1.925, 3.752]
Groups = 1.5	2.835	0.380	[2.158, 3.511]	3.351	0.387	[2.662, 4.041]
Groups = 2	2.345	0.532	[1.397, 3.292]	3.865	0.437	[3.104, 4.625]
Groups = 2.5	1.855	0.847	[0.345, 3.365]	4.378	0.600	[3.308, 5.447]
Groups = 3	1.366	1.204	[−0.780, 3.512]	4.891	0.827	[3.417, 6.364]

column of ones represents the constant (intercept). We can then express the vector of predicted values $\hat{\mathbf{y}}$ as

$$\hat{\mathbf{y}} = \mathbf{M_h}\hat{\boldsymbol{\beta}} \quad \text{where } \hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_x \\ \hat{\beta}_z \\ \hat{\beta}_{xz} \\ \hat{\beta}_0 \end{bmatrix}$$

As a consequence, $\mathbf{V}(\hat{\mathbf{y}}) = \mathbf{V}(\mathbf{M_h}\hat{\boldsymbol{\beta}})$. Since $\mathbf{M_h}$ is a matrix of values at which we set our independent variables, and since independent variables are fixed in repeated sampling under classical regression assumptions, the matrix $\mathbf{M_h}$ is a constant whereas $\hat{\boldsymbol{\beta}}$ is a random vector. Accordingly

$$\mathbf{V}(\hat{\mathbf{y}}) = \mathbf{V}(\mathbf{M_h}\hat{\boldsymbol{\beta}}) = \mathbf{M_h}\mathbf{V}\,(\hat{\boldsymbol{\beta}})\mathbf{M_h'}$$

where $\mathbf{V}(\hat{\boldsymbol{\beta}})$ is the variance-covariance matrix of the estimated coefficients.

The j diagonal elements in $\mathbf{V}(\hat{\mathbf{y}})$ correspond with the variances of the j predicted values of $\hat{\mathbf{y}}$ at various values included in $\mathbf{M_h}$. As before, we denote the estimate of $\mathbf{V}(\hat{\boldsymbol{\beta}})$ as $\widehat{\mathbf{V}(\hat{\boldsymbol{\beta}})}$.

Using our *Candidates* example, we can calculate the variance of the predicted values of y as follows. First, varying values of *Groups* in 0.5 intervals from 1 to 3, holding *Runoff* to 0, gives

$$\mathbf{M_h} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1.5 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2.5 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

The first column indicates the values of *Groups*, the second column indicates the values of *Runoff*, the third column indicates the values of *Groups* × *Runoff*, and the fourth column represents the values for the intercept. The estimated variances of the predicted numbers of candidates at these values are therefore given by

$$\widehat{\mathbf{V}(\hat{\mathbf{y}})} = \mathbf{M_h}\widehat{\mathbf{V}(\hat{\boldsymbol{\beta}})}\mathbf{M_h'} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1.5 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2.5 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.593 & 0.900 & -0.593 & -0.900 \\ 0.900 & 2.435 & -1.377 & -1.509 \\ -0.593 & -1.377 & 0.885 & 0.900 \\ -0.900 & -1.509 & 0.900 & 1.509 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 1.5 & 2 & 2.5 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which produces the following symmetric matrix:

$$\widehat{\mathbf{V}(\hat{\mathbf{y}})} = \begin{bmatrix} 0.302 & 0.149 & -0.005 & -0.159 & -0.312 \\ 0.149 & 0.143 & 0.138 & 0.133 & 0.128 \\ -0.005 & 0.138 & 0.281 & 0.424 & 0.567 \\ -0.159 & 0.133 & 0.424 & 0.715 & 1.007 \\ -0.312 & 0.128 & 0.567 & 1.007 & 1.446 \end{bmatrix}$$

The diagonal elements are $\widehat{V(\hat{y})}$ for the respective values of *Groups* when *Runoff* = 0. Statistical software or a basic spreadsheet program can make these matrix calculations simple to implement.

Predicted values are often more effectively displayed when graphed with confidence intervals, which can be constructed as $\hat{y} \pm t_{df,p} \sqrt{\widehat{V(\hat{y})}}$, where, as before, $t_{df,p}$ is the critical value in a t-distribution with df degrees of freedom that produces a p-value corresponding to half of the probability outside of the desired confidence interval. For example, lower and upper bounds of a 95 percent confidence interval will again come from $t_{df,p}$ of approximately 1.96 in large samples.

For this example, we calculate $\hat{y}$ along evenly spaced values of *Groups* from one to three, fixing *Runoff* first to zero and then to one. To calculate confidence intervals, we need to calculate the variances of these predicted values and to identify a desired level of confidence. Given our small sample, we again accept appreciable uncertainty, selecting a 90 percent confidence interval, implying a critical value of $t_{12,\alpha=0.10} = 1.782$. The upper bound and lower bound for the confidence intervals are therefore

Upper bound: $\hat{y} + 1.782 \times \sqrt{\widehat{V(\hat{y})}}$

Lower bound: $\hat{y} - 1.782 \times \sqrt{\widehat{V(\hat{y})}}$

For *Groups* = 1 and *Runoff* = 0, for example, $\widehat{V(\hat{y})} = 0.302$ as seen earlier, and so the 90 percent confidence interval is

Upper bound: $3.324 + 1.782 \times 0.302 = 4.304$

Lower bound: $3.324 - 1.782 \times 0.302 = 2.345$

Table 22 displays the confidence intervals calculated for the predicted values of the number of presidential candidates as *Groups* ranges from 1 to 3 in steps of 0.5, with *Runoff* fixed to 0 and to 1. Figure 18 graphs these predicted values and confidence intervals with *Groups* on the x-axis, the predicted values on the y-axis, and the value of *Runoff* fixed.

Figure 18 displays straight lines, indicating how the predicted number of candidates changes as *Groups* varies, in the presence and absence

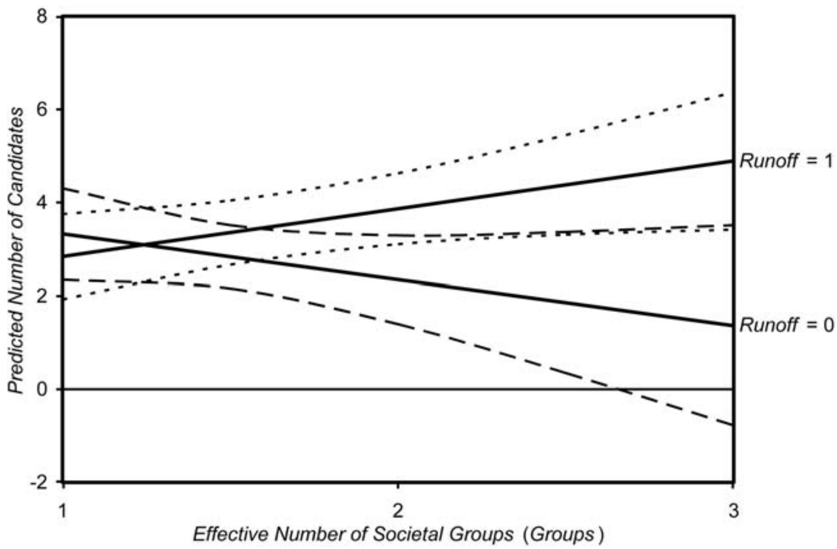

Fig. 18. ***Predicted Number of Candidates,*** **with 90 percent confidence intervals**

of a runoff system. The hourglass curves indicate the degree of certainty associated with each predicted value, $\hat{y}$. As with estimated effects, these predictions have greatest certainty around the mean of *Groups* and less certainty at more extreme and, especially, out-of-sample values.

Table 22 reinforces what we have already seen in this example: substantial overlap in the 90 percent confidence intervals for the predicted number of presidential candidates in the presence and absence of a runoff system when only one ethnic group exists but much less overlap in these confidence intervals at higher numbers of *Groups*. These results suggest that the impact of runoff systems on the number of candidates becomes more discernibly positive statistically as *Groups* increases.

Tables 23 and 24 provide confidence intervals for various predicted values in two of our other examples: the U.S. support for social welfare model and the baseline model of government duration (with just the interaction between the number of governing parties and parliamentary support). The results in table 23 are easily comprehensible, given that the interaction involves only two binary variables: *Female* and *Republican*. Table 23 shows the negligible difference in predicted social-welfare support among Democrats (*Republican* = 0) by gender, with the confidence intervals around those predicted values overlapping substantially (in fact, the confidence interval for male Democrats entirely encloses the confidence interval for female Democrats). We also see that social-welfare support among Republican males is appreciably lower than among Republican females. This gender gap among Republicans is statistically distinguishable from zero in that the confidence intervals for male and female Republicans do not overlap. The same information could also be presented graphically, but the simplicity of the table may recommend tabular form instead.

Table 24 adds 90 percent confidence intervals to the predicted values presented in table 6. It is less immediately interpretable, given the plethora of values that *NP* and *PS* can take. A graph may be the most effective means of presenting the predicted values and their associated confidence intervals in cases like this, as shown in figure 19.

For variables that enter nonlinearly—for instance, in the example where parliamentary support for government is quadratically related to government durability—the procedure previously outlined still obtains. Recall that figure 2 plotted estimated government duration as a quadratic function of parliamentary support. Accordingly, figure 10 adds 90 percent confidence intervals to the predicted value curve, using the results from the model in table 7 and calculating the confidence interval by

$$\begin{aligned}
\widehat{GD} &= \hat{\beta}_0 + \hat{\beta}_{ps}PS + \hat{\beta}_{ps^2}PS^2 \\
\widehat{V(\widehat{GD})} &= \widehat{V(\hat{\beta}_0 + \hat{\beta}_{ps}PS + \hat{\beta}_{ps^2}PS^2)} \\
&= \widehat{V(\hat{\beta}_0)} + PS^2 \times \widehat{V(\hat{\beta}_{ps})} + PS^4\widehat{V(\hat{\beta}_{ps^2})} + 2PS \times \widehat{C(\hat{\beta}_0,\hat{\beta}_{ps})} \\
&\quad + 2PS^2 \times \widehat{C(\hat{\beta}_0,\hat{\beta}_{ps^2})} + 2PS^3 \times \widehat{C(\hat{\beta}_{ps},\hat{\beta}_{ps^2})} \\
90\%\text{ c.i.} &= (\hat{\beta}_0 + \hat{\beta}_{ps}PS + \hat{\beta}_{ps^2}PS^2) \\
&\quad \pm 1.73 \begin{bmatrix} \widehat{V(\hat{\beta}_0)} + PS^2 \times \widehat{V(\hat{\beta}_{ps})} + PS^4\widehat{V(\hat{\beta}_{ps^2})} \\ + 2PS \times \widehat{C(\hat{\beta}_0,\hat{\beta}_{ps})} + 2PS^2 \times \widehat{C(\hat{\beta}_0,\hat{\beta}_{ps^2})} \\ + 2PS^3 \times \widehat{C(\hat{\beta}_{ps},\hat{\beta}_{ps^2})} \end{bmatrix}^{0.5}
\end{aligned}$$

In the log-transformed-*PS* model, we can also graph the predicted government duration, at selected values of *NP* and *PD*, along values of

TABLE 23. Confidence Intervals for Predicted *Support for Social Welfare*

	Republican = 0			*Republican* = 1		
	$\hat{y}$	$s.e.(\hat{y})$	95% Confidence Interval	$\hat{y}$	$s.e.(\hat{y})$	95% Confidence Interval
Female = 0	0.745	0.0110	[0.724, 0.767]	0.525	0.0110	[0.503, 0.546]
Female = 1	0.742	0.0094	[0.724, 0.760]	0.605	0.0113	[0.583, 0.627]

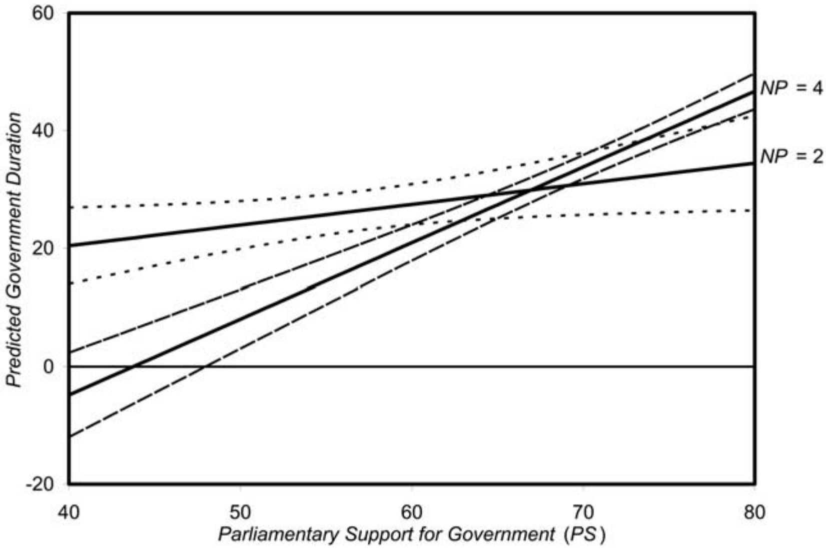

Fig. 19. Predicted *Government Duration,* with 90 percent confidence intervals

parliamentary support. The 90 percent confidence interval can be calculated around $\widehat{GD}$ as

$$\begin{aligned}
\widehat{GD} &= \hat{\beta}_0 + \hat{\beta}_{np}NP + \hat{\beta}_{\ln(ps)}\ln(PS) + \hat{\beta}_{np\ln(ps)}NP \times \ln(PS) \\
&\quad + \hat{\beta}_{pd}PD \\
\widehat{V(\widehat{GD})} &= \overline{V(\hat{\beta}_0 + \hat{\beta}_{np}NP + \hat{\beta}_{\ln(ps)}\ln(PS) + \hat{\beta}_{np\ln(ps)}NP \times \ln(PS)} \\
&\quad \overline{+ \hat{\beta}_{pd}PD)} \\
90\%\text{ c.i.} &= \widehat{GD} \pm 1.74\sqrt{\widehat{V(\widehat{GD})}}
\end{aligned}$$

The estimated government duration, calculated for $NP = 2$ and $NP = 4$, when $PD = 1$, and accompanying confidence intervals appear in figure 20.

TABLE 24. Confidence Intervals for Predicted *Government Duration*

	$NP = 1$		$NP = 2$		$NP = 3$		$NP = 4$	
	$\hat{y}$	90% Confidence Interval	$\hat{y}$	90% Confidence Interval	$\hat{y}$	90% Confidence Interval	$\hat{y}$	90% Confidence Interval
$PS = 40$	33.05	[23.87, 42.23]	20.42	[13.94, 26.90]	7.79	[−2.37, 17.96]	−4.84	[−21.21, 11.54]
$PS = 50$	31.87	[26.59, 37.15]	23.93	[19.86, 28.00]	15.99	[9.28, 22.69]	8.05	[−2.58, 18.67]
$PS = 60$	30.70	[25.87, 35.53]	27.44	[23.99, 30.89]	24.18	[19.92, 28.45]	20.93	[14.43, 27.43]
$PS = 70$	29.52	[21.11, 37.93]	30.95	[25.67, 36.23]	32.38	[27.58, 37.17]	33.81	[26.33, 41.29]
$PS = 80$	28.34	[15.31, 41.38]	34.46	[26.42, 42.50]	40.57	[32.86, 48.28]	46.69	[34.27, 59.10]

Note: Predicted values are calculated at given values, setting $PD = 1$.

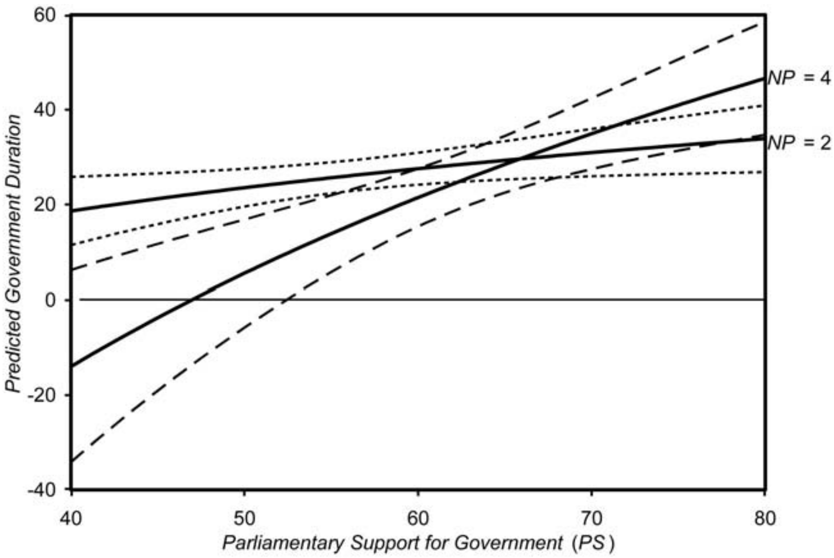

Fig. 20. Predicted *Government Duration,* log-transformation interactive model, with 90 percent confidence intervals

Presentation of Differences of Predicted Values

Predicted values display how variation along some range of an independent variable, x, affects the level of the dependent variable, conditional upon a third independent variable, z. Researchers may sometimes wish to present the estimated effects of discrete changes rather than marginal changes of independent variables involved in interaction terms: $\Delta y/\Delta x$ rather than $\partial y/\partial x$. For example, one might want to plot the estimated effect of some substantively motivated counterfactual increase or decrease in an independent variable, say, of a 10 percent increase in parliamentary support; or of a unit change in binary indicators like gender, partisanship, runoff, or party discipline; or of a change from the level of some well-known exemplar to another (e.g., from the average number of governing parties in the United Kingdom, 1, to that of the Netherlands, 3.3). Provided that the variables involved in the estimated conditional effect enter only linearly, as in all of our examples except those using the square or natural log of parliamentary support (tables 7 and 8), doing so requires only a very simple extension of our preceding discussion of presenting marginal effects.

In regression models where the independent variables enter only linearly or linear interactively, the estimated marginal effect of any variable is equal to the estimated effect of a unit increase in that variable. In a linear-interaction model involving x, z, and xz, for example, $\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z$ and $\Delta\hat{y}/\Delta x = \hat{\beta}_x\Delta x + \hat{\beta}_{xz}(\Delta x)z$, which gives $\Delta\hat{y}/\Delta x = \hat{\beta}_x + \hat{\beta}_{xz}z$ for $\Delta x = 1$. Of course, their estimated standard errors are also identical. Thus, figures 4–9 and 13–17 all give the estimated effects of a unit increase as well as the estimated slope (or effect of a marginal increase) of their respective independent variables in their respective models.[52] More generally, if we wanted to present the effect of some discrete change other than $\Delta x = 1$ in a linear-interaction model, we need only replace the marginal effect, $\partial\hat{y}/\partial x = \hat{\beta}_x + \hat{\beta}_{xz}z$, with that of the change, $\Delta\hat{y}/\Delta x = \hat{\beta}_x\Delta x + \hat{\beta}_{xz}(\Delta x)z$, which amounts simply to multiplying the marginal effect by Δx: $\Delta\hat{y}/\Delta x = \Delta x(\hat{\beta}_x + \hat{\beta}_{xz}z)$. To estimate standard errors for confidence intervals around differences in predicted values, we apply the usual variance formula: $\widehat{V(\Delta\hat{y}/\Delta x)} = (\Delta x)^2\,\widehat{V(\hat{\beta}_x)} + z^2(\Delta x)^2\,\widehat{V(\hat{\beta}_{xz})} + 2z(\Delta x)^2\,\widehat{C(\hat{\beta}_x,\hat{\beta}_{xz})}$, that is, we multiply the estimated variance of the estimated marginal effect, $\widehat{V(\Delta\hat{y}/\Delta x)} = \widehat{V(\hat{\beta}_x)} + z^2\widehat{V(\hat{\beta}_{xz})} + 2z\widehat{C(\hat{\beta}_x,\hat{\beta}_{xz})}$, by $(\Delta x)^2$.

We can use the estimates in table 22 to determine the effect of a *Runoff* at various values of *Group* using the difference method simply by subtracting the first from the fourth column, that is, ($\hat{y}$ | *Groups*, *Runoff* = 1) − ($\hat{y}$ | *Groups*, *Runoff* = 0). Recall that the case of $(x_c - x_a) = 1$ produces exactly the same results as the derivative method; the differences in predicted values between systems with runoffs and without runoffs, at given values of *Groups* (and the corresponding estimates of uncertainty around those differences in predicted values), appear in table 15.

More generally, for binary variables like our *Runoff*, *Gender*, *Partisanship*, or *Party Discipline*, the only discrete changes meriting consideration are unit increases or decreases, $\Delta x = \pm 1$, and so the estimated marginal effects and confidence intervals plotted in figures 4, 5, 7, 15, and 17 are all identical to the estimated conditional effects of and confidence intervals for a positive switch in the value of that binary indicator. Similarly, the estimated marginal effects of *Groups*, *NP*, and *PS* and confidence intervals plotted in figures 6, 8, 9, 13, 14, and 16 are all identical to the estimated effects and confidence intervals for unit increases in those (nonbinary) variables. If we had wanted to present the estimated effects of, say, a 10 percent rather than a unit (1 percent) increase in parliamentary support, for example, we would simply have multiplied the

52. In fact, in our "log-transformed" model of government duration from table 8, the effect of a unit increase in the number of governing parties, which itself enters the model linearly, is linear in the natural log of parliamentary support, and so, had ln(*PS*) been the *x*-axis of figure 12, the same would apply for that presentation.

conditional effect line in figures 9, 14, and 16 by 10, the variance in the formula for the confidence intervals associated with those lines by 10^2, and relabeled the figure as "Effect of a 10% Increase in *Parliamentary Support . . .*"

Graphs of the estimated effects of discrete changes would therefore simply rescale the marginal-effect graphs already shown. We can demonstrate this formally for the standard linear-interaction model as follows. The difference between $\hat{y}_a$ and $\hat{y}_c$, that is, $\hat{y}$ at $x = x_a$ subtracted from $\hat{y}$ at $x = x_c$, is

$$\begin{aligned}\hat{y}_c - \hat{y}_a &= \hat{\gamma}_0 + \hat{\beta}_x x_c + \hat{\beta}_z z_0 + \hat{\beta}_{xz} x_c z_0 - (\hat{\gamma}_0 + \hat{\beta}_x x_a + \hat{\beta}_z z_0 + \hat{\beta}_{xz} x_a z_0)\\ &= \hat{\beta}_x (x_c - x_a) + \hat{\beta}_{xz} z_0 (x_c - x_a)\\ &= (x_c - x_a)(\hat{\beta}_x + \hat{\beta}_{xz} z_0)\end{aligned}$$

The variance of that difference is then

$$\begin{aligned}V(\hat{y}_c - \hat{y}_a) &= V[(x_c - x_a)\,(\hat{\beta}_x + \hat{\beta}_{xz} z_0)]\\ &= (x_c - x_a)^2 V[(\hat{\beta}_x + \hat{\beta}_{xz} z_0)]\\ &= (x_c - x_a)^2 [V(\hat{\beta}_x) + z_0^2 V(\hat{\beta}_{xz}) + 2 z_0 C(\hat{\beta}_x, \hat{\beta}_{xz})]\end{aligned}$$

So, in the case of $(x_c - x_a) = 1$, we have exactly the same results as the derivative method, and in the case of $(x_c - x_a) = \Delta x$ we have the same results rescaled multiplicatively by Δx.

We could also tabulate and/or graph the difference in predicted values as the number of societal groups changes, by one unit (say, from *Groups* = 1 to *Groups* = 2 or, equivalently, from *Groups* = 2 to *Groups* = 3), or by two units (from *Groups* = 1 [the sample minimum] to *Groups* = 3 [just above the sample maximum]), by the presence or absence of a runoff. The differences in predicted values that correspond with a one-unit and a two-unit shift in *Groups*, by *Runoff*, appear in table 25. We could also present a graph containing the difference in predicted values associated with a two-unit shift in *Groups*, but since

TABLE 25. Confidence Intervals for Differences in Predicted *Number of Candidates*

	Runoff = 0			*Runoff* = 1		
	$\hat{y}_c - \hat{y}_a$	$s.e.(\hat{y}_c - \hat{y}_a)$	90% Confidence Interval	$\hat{y}_c - \hat{y}_a$	$s.e.(\hat{y}_c - \hat{y}_a)$	90% Confidence Interval
Δ*Groups* = 1	−0.979	0.770	[−2.352, 0.394]	1.026	0.540	[0.064, 1.988]
Δ*Groups* = 2	−1.958	1.541	[−4.704, 0.787]	2.052	1.079	[0.129, 3.976]

there are only two points to be graphed, a table like table 25 is just as informative.

When variables enter the regression models nonlinearly, however, as in the quadratic- and log-transformed-*PS* models of tables 7 and 8, the effect of a discrete change from one value of x to another can be quite different than the effect of a marginal (i.e., infinitesimal) change at that x. That is, except for straight lines, *derivatives* and *slopes* differ from *differences*. In figure 2, for example, the marginal effect of parliamentary support at $PS = 50$ (i.e., the derivative or slope at that point) is $\partial \widehat{GD}/\partial PS = \hat{\beta}_{ps} + 2\hat{\beta}_{ps^2}\, 50 = \hat{\beta}_{ps} + 100 \times \hat{\beta}_{ps^2}$. The effect of a unit change from $PS =$ 50 to 51 would be $\Delta\widehat{GD}/\Delta PS = (\hat{\beta}_{ps}\, 51 + \hat{\beta}_{ps^2}\, 51^2) - (\hat{\beta}_{ps}\, 50 + \hat{\beta}_{ps^2}\, 50^2) = \hat{\beta}_{ps} + \hat{\beta}_{ps^2}\, (51^2 - 50^2) = \hat{\beta}_{ps} + \hat{\beta}_{ps^2}\, 101$.[53] The estimated variances would differ accordingly. The estimated effect of a 10 percent increase from $PS = 45$ percent to 55 percent, $\Delta\widehat{GD}/\Delta PS = (\hat{\beta}_{ps}\, 55 + \hat{\beta}_{ps^2}\, 55^2) - (\hat{\beta}_{ps}\, 45 + \hat{\beta}_{ps^2}\, 45^2) = \hat{\beta}_{ps}\, 10 + \hat{\beta}_{ps^2}\, 1{,}000$, is likewise not equal to the slope at $PS = 45$, $\partial\widehat{GD}/\partial PS = \hat{\beta}_{ps} + 90\, \hat{\beta}_{ps^2}$, and their standard errors differ also.

Similarly in the log-transformed-*PS* model, the effects of discrete changes in parliamentary support depend not only on the number of governing parties but on the magnitudes and values of those changes in *PS*. The estimated effect of a 10 percent increase, from $PS = 45$ percent to $PS = 55$ percent, and its standard error would be

$$
\begin{aligned}
(\widehat{GD}|PS = 55) - (\widehat{GD}|PS = 45) &= \hat{\beta}_{\ln(ps)}\, (\ln 55 - \ln 45) + \hat{\beta}_{np\ln(ps)} NP \\
&\quad \times (\ln 55 - \ln 45) \\
\widehat{V\big((\widehat{GD}|PS = 55) - (\widehat{GD}|PS = 45)\big)} &= (\ln 55 - \ln 45)^2 \\
&\quad \times \big(\widehat{V(\hat{\beta}_{\ln(ps)})} + NP^2 \times \widehat{V(\hat{\beta}_{np\ln(ps)})} \\
&\quad + 2NP \times \widehat{C(\hat{\beta}_{\ln(ps)}, \hat{\beta}_{np\ln(ps)})}\big)
\end{aligned}
$$

As these two examples illustrate, the conditional effect of a discrete change in an independent variable that enters an interaction model nonlinearly depends not only on the values of the variables with which it interacts but also on the magnitude of the change and from what starting point. We would not, therefore, generally recommend graphing but rather recommend tabulating sets of estimates like these for consideration and discussion.

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53. We can focus only on the terms that would involve Δx because the rest of the equation drops from these differences.

Distinguishing between Conditional Effects and Predicted Values

How do tables and graphs of conditional effects and those of predicted values differ? Both reveal information about the relation of x to y and how this relationship changes as z varies but in slightly different ways. Graphs and tables of derivatives of or differences in predicted values show directly how the *effect* of x on y changes as z changes. Graphs and tables of predicted values show how the *level* of $\hat{y}$, that is, the prediction for y, changes as x changes, at particular levels of z. By comparing several of these predicted levels, one can also grasp the effects of x or of z on y and how they change as the other variable changes, but, in predicted-level tables and figures, the comparison of effects is less direct and the uncertainties related refer to individual predictions and not to these differences, that is, not to the effects. Selection of one type of table or graph over the other therefore largely depends on the researcher's presentational goals. Either method can effectively convey the substantive results from empirical models involving interactive terms; we stress, however, that either sort of table or graph should incorporate measures of uncertainty into its presentation.